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Introductory Survey for Recent Restrictive Approximations and Applications of Solutions of Initial Boundary Value Probles for PDE

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Abstract. This research dealt with the restrictive Pade` approximations and restrictive Taylor's approximations, this paper contains the definitions of restrictive approximations and computing applications to many problems of initial and boundary value problems of systems of partial differential equations of parabolic and hyperbolic types, linear and nonlinear problems also that of variable coefficients and singularly perturbed problems of one dimension or more. It used restrictive Pade` approximations and Taylor's approximations which applied on physical phenomena such as convection-diffusion equations. The suggested method is more efficient and more accurate than many known methods, also it is of low cost. The local truncation error in our method is less than that for the classical known methods, in our method it varies from I0-8 to 10-15 while in the classical known methods it varies from 10-2 to10-9. The efficiency of our method is better than that for Douglass and Rachford (D.R.) method because of the ratio of the number of systems to be solved between D.R. method and this method varies from 2: 1 to 10: 1, also their method is more fast than D.R. method because the ratio of the executed time of their suggested method to D.R. method varies from 1: 2 to 1: 15 the suggested method is considered as an acceleration technique for the classical finite difference methods (appending in Elbarbary, 2001).

Key words: Introductory Survey, Recent Restrictive Approximations, Applications, Solutions of Initial Boundary Value.

Introduction

This research dealt with the restrictive Pade` approximations and restrictive Taylor's approximations, this paper contains the definitions of restrictive approximations and computing applications to many problems of initial and boundary value problems of systems of partial differential equations of parabolic and hyperbolic types, linear and nonlinear problems also that of variable coefficients and singularly perturbed problems of one dimension or more, many of the data and results **depended** in **(**Elbarbary, 2001)

1. Approximations and Restrictive Approximations

1.1 Pade` and Taylor Approximations of Scalar Functions

Padé approximations known as PAI, PAII and PAIII has been developed in theoretical physics and mechanics to solve a great number of problems

(Baker G and Morris P, 1981). Several types of the approximations occurred, the famous one is constructed by segment of Taylor's series of a studied function, the use of interpolation approximations drawn through given n-points and using the least squares method which construct the rational function from a given data.

Let f(x) be has a Maclaurin series

$$f(x) = \sum_{i=0}^{\infty} c_i x^i \qquad (.1.1)$$

where $c_i = \dfrac{f^{(i)}(0)}{i!}$

The Padé Approximation (PA) of the function f(x) is denoted by $[M, N]_{f(x)}(x)$, where

$$[M,N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i x^i}{1+\sum_{i=1}^{N} b_i x^i} \qquad (1.2)$$

Such that $a_M \neq 0$ and $b_N \neq 0$, M and N are positive integers such that

$$f(x) = [M, N]_{f(x)}(x) + O(x^{M+N+1}).$$

The Pade` approximants to e^{θ}, θ real (**Smith G**, 1985):

Assume that e^{θ} is approximated by $(1+p_1\theta)/(1+q_1\theta)$, where p_1 and q_1 are constants. The determination of p_1 and q_1 requires two equations, which will come from the coefficients of θ, θ^2 and so the leading error term will be of order θ^3. Hence

$$e^{\theta} \equiv \frac{1+p_1\theta}{1+q_1\theta} + c_3\theta^3 + c_4\theta^4 + \cdots$$

Therefore,

$$(1+q_1\theta)\left(1+\theta+\frac{1}{2}\theta^2+\frac{1}{6}\theta^3+\cdots\right) \equiv 1+p_1\theta+(1+q_1\theta)(c_3\theta^3+c_4\theta^4+\cdots).$$

Hence,

$$(1+q_1-p_1)\theta + \left(\frac{1}{2}+q_1\right)\theta^2 + \left(\frac{1}{6}+\frac{1}{2}q_1-c_3\right)\theta^3 + \text{ higher order terms} \equiv 0.$$

This is satisfied uniquely to terms of order three by

$$p_1 = \frac{1}{2}, \quad q_1 = -\frac{1}{2} \quad and \quad c_3 = -\frac{1}{12}.$$

The rational approximation $\dfrac{1+0.5\,\theta}{1-0.5\,\theta}$ is called the (1,1) Pade` approximation of order 2 to $exp\,\theta$ and has a leading error term of order *3*.

The following table gives the first four Pade` approximants to e^{θ} and their leading error terms:

(S,T)	$R_{S,T}(\theta)$	Principal error term
(0,1)	$1+\theta$	$0.5\theta^2$
(0,2)	$1+\theta+0.5\,\theta^2$	$\frac{1}{6}\theta^3$
(1,0)	$\frac{1}{1-\theta}$	$-\frac{1}{2}\theta^2$
(1,1)	$\frac{1+0.5\,\theta}{1-0.5\,\theta}$	$-\frac{1}{12}\theta^3$

In general, it is possible to approximate $exp\,\theta$ by

$$e^{\theta} = \frac{1+p_1\theta+p_2\theta^2+\cdots+p_T\theta^T}{1+q_1\theta+q_2\theta^2+\cdots+q_S\theta^S} + c_{S+T+1}\theta^{S+T+1} + O(\theta^{S+T+2}),$$

where c_{S+T+1} is a constant. The rational function

$$R_{S,T}(\theta) = \frac{1+p_1\theta+\cdots+p_T\theta^T}{1+q_1\theta+\cdots+q_S\theta^S} = \frac{P_T(\theta)}{Q_S(\theta)}$$

Where $(b_1, b_2, \dots, b_T)$, $(q_1, q_2, \dots, q_S)$ can be uniquely determined and T, S is given, (S, T) is called the Pade` or [S, T] approximant of order (S+T) to e^{θ}.

The classical implicit approximation:

The (1,0) Pade` approximant approximates $V(t+k) = \{exp(Ka)\}V(t)$ by

$$U(t+k)=(I-kA)^{-1}u(t).$$

Premultiplication of both sides by the matrix $(I-kA)$ yields

$$(I-kA)u(t_j+k)=u(t_j),\ \ j=0,1,2,\ldots,$$

where $u(t_j+k)=[u_{1,j+1},u_{2,j+1},\ldots,u_{N-1,j+1}]^T$ and matrix A is defined as

$$A=\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & & . & & & \\ & & & . & & \\ & & & & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

In details, for zero boundary –conditions,

$$\begin{pmatrix} (1+2r) & -r & & & \\ -r & (1+2r) & -r & & \\ & & & & \\ & & & (1+2r) & -r \\ & & & -r & (1+2r) \end{pmatrix} \times \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-2,j+1} \\ u_{N-1,j+1} \end{pmatrix} = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-2,j} \\ u_{N-1,j} \end{pmatrix}$$

The i^{th} equation gives the implicit or backward-difference scheme

$$-ru_{i-1,j+1}+(1+2r)u_{i,j+1}-ru_{i+1,j+1}=u_{i,j},\quad i=1(1)N-1.$$

This is unconditionally stable for all $r=k/h^2>0$.The leading error terms are of order h^2 in x because of the central-difference approximation to $\frac{\partial^2 u}{\partial x^2}$ and of order k in t.(The leading error term of the (1,0) Pade` approximant to $exp(kA)$ is $O(k^2)$) The method is said to be first order accurate in t .

1.2 Restrictive Approximations of a Scalar Function:

The Restrictive Padé Approximation (RPA) of the Function f(x)

Restrictive Pade` approximation of a function f(x) is constructed in

(Hassan, Ismail and Elbarbary. 1995) with parameters ε_i's to be determined, which if reduces to zero the classical Padé approximation [2] is obtained. The advantage is that it has zero error at certain points and consequently relatively small in between.

The restrictive Padé approximation (RPA) of the function f(x) is

$$RPA[M,\alpha,N]_{f(x)}(x)=\frac{\sum_{i=0}^{M}a_i x^i+\sum_{i=M+1}^{M+\alpha}\varepsilon_{i-M}x^i}{1+\sum_{i=1}^{N}b_i x^i} \qquad (1.3)$$

where $\varepsilon_j=0\ \forall j>\alpha$ and $\alpha=0(1)N$ i.e., the positive integer α does not exceed the degree of the numerator, then

$$f(x)=RPA[M,\alpha,N]_{f(x)}(x)+O(x^{M+N+1}) \quad (1.4)$$

and let f(x) have a Maclaurin series (1.1.1)

Then

$$\left(\sum_{i=0}^{\infty}c_i x^i\right)\left(1+\sum_{i=0}^{N}b_i x^i\right)-\left(\sum_{i=0}^{M}a_i x^i\right)-\left(\sum_{i=M+1}^{M+N}\varepsilon_{i-M}x^i\right)=O\left(x^{M+N+1}\right) \qquad (1.5)$$

The vanishing of the first $(M+N+1)$ powers of x on the left-hand side of (1.5) implies the system of $(M+N+1)$ equations:

$$a_r=c_r+\sum_{j=1}^{M}c_{r-j}b_j\ ,\ r=0(1)M\ ; \qquad (b_j=0 \text{ if } j>M)$$

$$c_{M+N-s}+\sum_{j=1}^{N}c_{M+N-j-s}b_j=\varepsilon_{N-s}\ ,\ s=0(1); \qquad (c_j=0 \text{ if } j<0)$$

S.t. we can determine the coefficient a_i and b_i as a function of ε, $i = 1(1)\alpha$, and let $[M,\alpha, N]_{f(x)}(x_j)$

$= f(x_j)$, $j = 1(1)\alpha$ when this system of $(M + N + \alpha +1)$ linear equations in $(M + N + \alpha +1)$ unknowns has a solution, it leads to the desired approximation (1.3). If $\varepsilon_i=0, \forall i =1(1)\alpha$ in (1.5) we get the classical Padé approximation (1.2)

The RPA can be arranged in a doubly infinite array, each entry of which represents $[M, \alpha, N]_{f(x)}(x)$, $\alpha \le N$ for $f(x)$.

The case $\alpha = 0$ gives the classical Padé approximation table.

The case $\alpha = 1$ gives, for examples, the following selected elements of RPA:

$$[0, 1, 1]_{f(x)}(x) = \frac{a_0 + \varepsilon_1 x}{1 + b_1 x}$$

where

$$a_o = c_o \ , \ b_1 = \frac{\varepsilon_1 - c_2}{c_0},$$

$$[1, 1, 1]_{f(x)}(x) = \frac{a_0 + a_1 x + \varepsilon_1 x^2}{1 + b_1 x}$$

where $a_o = c_o$, $a_1 = c_1 + \dfrac{c_0(\varepsilon_1 - c_2)}{c_1}$, and $b_1 = \dfrac{\varepsilon_1 - c_2}{c_1}$,

$$[1, 1, 2]_{f(x)}(x) = \frac{a_0 + a_1 x + \varepsilon_1 x^2}{1 + b_1 x + b_2 x^2}$$

where $a_o= c_o$, $a_1= c_1 + \dfrac{c_0(c_1c_2 - c_0c_3 - c_1\varepsilon_1)}{c_0c_2 - c_1^2}$, $b_1= \dfrac{c_1c_2 - c_0c_3 - c_1\varepsilon_1}{c_0c_2 - c_1^2}$ and $b_2 = \dfrac{c_1c_3 - c_2^2 - c_2\varepsilon_1}{c_0c_2 - c_1^2}$.

The case $\alpha = 2$, gives, for examples, the following selected elements of RPA:

$$[0, 2, 2]_{f(x)}(x) = \frac{a_0 + \varepsilon_1 x + \varepsilon_2 x^2}{1 + b_1 x + b_2 x^2}$$

where $a_o = c_o$, $b_1 = \dfrac{\varepsilon_1 - c_1}{c_0}$, and $b_2 = \dfrac{c_1^2 - c_0c_2 - c_1\varepsilon_1 + c_0\varepsilon_2}{c_0^2}$, and

$$[1, 2, 2]_{f(x)}(x) = \frac{a_0 + a_1 x + \varepsilon_1 x^2 + \varepsilon_2 x^3}{1 + b_1 x + b_2 x^2}$$

where

$a_o = c_o$, $a_1 = \dfrac{2c_0c_1c_2 - c_0^2c_3 - c_0c_1\varepsilon_1 + c_0^2\varepsilon_2 - c_1^3)}{c_0c_2 - c_1^2}$, $b_1 = \dfrac{c_1c_2 - c_0c_3 - c_1\varepsilon_1 + c_0\varepsilon_2}{c_0c_2 - c_1^2}$ and $b_2 =$ $\dfrac{c_1c_3 - c_2^2 - c_2\varepsilon_1 - c_1\varepsilon_2}{c_0c_2 - c_1^2}$ and so on.

For **the *Local Truncation Error (LTE)* form of RPA** of the function $f(x)$ of an $(n+1)^{th}$ derivative, it can be shown that for every argument $\bar{x}$ there exists a number η in the smallest interval I containing the set of points $\{x_o, x_1, x_2, x_3, \ldots, x_\alpha, \bar{x}\}$, such that

$$R_{M,\alpha,N}(\bar{x}) = f(\bar{x}) - [M,\alpha, N]_{f(x)}(\bar{x}) = \frac{\pi_{\alpha+1}(\bar{x})}{(\alpha+1)!}\left(R_{M,\alpha,N}(\eta)\right)^{(\alpha+1)},$$

where $\pi_{\alpha+1}(x)=x(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_\alpha)$ and $R_{M,\alpha,N}$ is the local truncation error for RPA.

A Restrictive Taylor `s Approximation of the Function f(x) (Hassan, Ismail and Elbarbary.2001):

A new approach for an explicit method to solve the parabolic partial differential equation is developed. This approach will exhibit several advantages features: Highly accurate, fast, and the absolute error still very small whatever the exact solution is too large. The computational results, which are presented, show that computing time and memory space are saved with more accuracy. Stability conditions are obtained and utilized in numerical computations.

One of the strategies for approximation of the function *f(x)* is by using Taylor's expansion, which approximate *f(x)* by a polynomial of degree n and its truncation error is of order $n+1$.

$$f(x) = P_{n,f(x)}(x) + R_{n+1}(x).$$

In a more extensive form *f(x)* can be written out as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \quad (1.6)$$

where

$$R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \quad (1.7)$$

and ξ lies between a and *x*.

In the paper (Hassan, Ismail and Elbarbary.2001) the authors constructed a restrictive type of Taylor's expansion of the function *f(x)* at the point a, with parameter to be determined such that the error equal to zero at certain point x_o. If the parameter reduces to one, we get the classical Taylor's expansion at the point a. The advantage is that it has exact value at the points *a* and x_o and relatively small in between.

Consider a function *f(x)* defined in a neighborhood of the point *a*, and it has derivatives up to order *(n+1)* in this neighborhood.

Then

$$RT_{n,f(x)}(x,a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{\varepsilon f^{(n)}(a)}{n!}(x-a)^n \quad (1.8)$$

Now, $RT_{n,\,f(x)}(x_o) = f(x_o)$.It means that the considered approximation is exact at two points *a* and x_o.

Putting

$$f(x) = RT_{n,f(x)}(x) + R_{n+1}(x), \quad (1.9)$$

where $R_{n+1}(x)$ is the remainder term of the restrictive Taylor's series.

In the following Theorem the remainder term $R_{n+1}(x)$ can be expressed in terms of ε, n^{th} and $(n+1)^{th}$ derivatives of the function *f(x)* at a point ξ lies between *a* and *x*.

Consider the function *f(x)*; $f(x) \in C^{n+1}$, $x \in I$, *I* is the neighborhood of a point *a*. The error for approximation estimated by $R_{n+1}(x)$ is given by the formula (1.9), for which

$$R_{n+1}(x) = \frac{\varepsilon(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) - \frac{n(\varepsilon-1)(x-a)^{n+1}}{(x-\xi)(n+1)!} f^{(n)}(\xi)$$

where $\xi \in [a,x]$ and ε is a restrictive parameter (Hassan, Ismail and Elbarbary.2001)

Note: $\varepsilon = 1$ gives the classical Taylor's expansion (1.1.6) and remainder term (1.7) and

$$\varepsilon = \frac{-nf^{(n)}(\xi)}{(x_o-\xi) f^{(n+1)}(\xi) - nf^{(n)}(\xi)}$$

gives the exact value at the point x_o, in which $R_{n+1}(x_o) = 0$.

1.3 Restrictive Padé and Restrictive Taylor's Approximation of Matrix *Functions:*

Padé Approximation of Matrix Functions:

It can be proved the Pade` approximation of the exponential matrix in many references e.g. (Smith G, 1985) when we study methods of finite differences as in that:

$$PA_{e^{Ar}}[1,1] = \left(I - \frac{1}{2}A\,r\right)^{-1}\left(I + \frac{1}{2}A\,r\right). \quad (1.10)$$

Restrictive Pade` Approximation of the Exponential Matrix:

The restrictive Pade` approximation is used to approximate the exponential matrix *exp(rA)*. The advantage is that it has the exact value at certain r

The exponential matrix *exp(rA)* can be formally defined by the convergent power series:

$$Exp\,(rA) = I + rA + \frac{r^2}{2!}A^2 + \cdots = \sum_{n=0}^{\infty}\frac{r^2}{n!}A^n, \; A^o = I$$

where *A* is an $(N-1) \times (N-1)$ matrix.

In the case of restrictive Pade` approximation of single function the term ε_i in equation (1.3) can be reduced to the square restrictive matrix $\in_{(N-1)\times(N-1)}$.

In the case of restrictive Pade` approximation for matrix function, where

$$\in = \begin{pmatrix} \varepsilon_1 & \varepsilon_1 & & & & 0 \\ \varepsilon_2 & \varepsilon_2 & \varepsilon_2 & & & \\ & \varepsilon_3 & \varepsilon_3 & \varepsilon_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \varepsilon_{N-2} & \varepsilon_{N-2} & \varepsilon_{N-2} \\ 0 & & & & \varepsilon_{N-1} & \varepsilon_{N-1} \end{pmatrix}$$

Without loss of generality, the authors chose $\varepsilon_{i,j}$ to be constant in each row i.e., $\varepsilon_{i,i-1} = \varepsilon_{i,i} = \varepsilon_{i,i+1}$ and $\varepsilon_{i,j} = 0$ otherwise, for example

$$RPA[0,1,1]_{exp(rA)}(r) = \left(I + (\in - \frac{1}{2}A)r\right)^{-1}\left(I + (\in + \frac{1}{2}A)r\right). \qquad (1.11)$$

Restrictive Taylor's Approximation of the Exponential Matrix (Hassan, Ismail and Elbarbary.2001):

Consider the exponential matrix *exp(xA)* in the form (1.10).

In the case of restrictive Taylor's approximation of single function the term ε in (1.1.8) can be reduced to the square restrictive matrix Υ in the case of restrictive Taylor's approximation for matrix function, where

$$\Upsilon = \begin{pmatrix} \varepsilon_1 & & & & & 0 \\ & \varepsilon_2 & & & & \\ & & \varepsilon_3 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & \varepsilon_{N-2} & \\ 0 & & & & & \varepsilon_{N-1} \end{pmatrix}_{(N-1)\times(N-1)}$$

for example $RT_{\exp(Ax)}(0,1)(x) = I + x\Upsilon A.$

1.4 Uniqueness and Solvability of Restrictive Approximations

Ismail (2004) introduced a general theory for solvability and uniqueness of the restrictive Pade` approximation (RPA) and restrictive Taylor's approximation (RTA). A survey of the individual necessary and sufficient solvability and uniqueness conditions for fifteen examples of these approximations are given.

Solvability and Uniqueness Condition for RPA:

The author introduced the solvability and uniqueness condition for restrictive Pade` approximation (1.3) - (1.5) for the sufficiently smooth function *f(x)* is uniquely solvable for $\pi_{i=1}^{\alpha} x_i \neq 0$ if the following $N\times N$ restrictive determinant $|D| \equiv \Delta \neq 0$:

$$\Delta = \begin{vmatrix} c_{M+\alpha} & c_{M+\alpha-1} & \cdots & c_{M+\alpha-N+1} \\ c_{M+\alpha+1} & c_{M+\alpha} & \cdots & c_{M+\alpha-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M+N-1} & c_{M+N-2} & \cdots & c_M \\ R_{M+\alpha-1}(x_1) & x_1 R_{M+\alpha-2}(x_1) & \cdots & x_1^{N-1} R_{M+\alpha-N}(x_1) \\ R_{M+\alpha-1}(x_2) & x_2 R_{M+\alpha-2}(x_2) & \cdots & x_2^{N-1} R_{M+\alpha-N}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{M+\alpha-1}(x_\alpha) & x_\alpha R_{M+\alpha-2}(x_\alpha) & \cdots & x_\alpha^{N-1} R_{M+\alpha-N}(x_\alpha) \end{vmatrix}$$

where:

i)

$$R_n(x) = P_n(x) - f(x),\ P_n(x) = \sum_{i=0}^{n} c_i x^i\ ,\ c_i = \frac{1}{i!} f^{(i)}(0),$$

ii) The c_n*'s* in the first *(N-α)*-rows and $R_n(x_i)$'s in the last α-rows such that n decreases toward the right. While the right remaining elements in each row must be zeros. When the last nonzero terms are c_o or $R_o(x)$ respectively. *i.e* $k < 0 \Rightarrow c_k \equiv 0,\ P_k(x_j) \equiv -f(x_j),\ j = 1(1)\alpha$,

iii) In the case of the absence of the mentioned α-restrictions, it is clear the reduction of this condition to the well-known C-table *C(M,N)* in G.A.Bakerwhich discuss the solvability condition for Pade` appro- ximations. Similarly it can be defined “The normal Restrictive Pade` Box” of the tables $C(M,\alpha, N) \equiv \Delta$ if none of these determinants vanish.

Solvability and Uniqueness Condition for RTA:

The restrictive Taylor's approximation is a polynomial satisfying the conditions:

$$RT_{f(x)}(M,\alpha)(x) = \sum_{i=o}^{M} a_i x^i + \sum_{i=1}^{\alpha} \varepsilon_i\, x^{M+i}\ ,\ x \in R \qquad (1.12)$$

Such that :

$$RT_{f(x)}(M,\alpha)(x) - f(x) = O(x^{M+1}), \qquad (1.13)$$

$$RT_{f(x)}(M,\alpha)(x_i) = f(x_i)\ \ \forall x_i \neq 0,\ i=1(1)\alpha. \qquad (1.14)$$

where

i) M is a nonnegative integer

ii) α is a positive integer

iii) $f(x) = \sum_{i=o}^{\infty} c_i (x - x_o)^i$ for simplicity $x_o=0 \Rightarrow c_i = \frac{1}{i!} f^{(i)}(0)$ i.e. *f(x)* is sufficiently smooth function at $x = 0$.

Uniqueness and solvability of restrictive Taylor’s approximation:

Restrictive Taylor’s approximation (1.12)-(1.14) is uniquely solvable if:

$$\forall i, j = 1(1)\alpha;\ x_i \neq 0\ ,\ x_i \neq x_j\ \ \forall i \neq j.$$

(Note that putting $N = 0$ for restrictive Pade` approximation (1.3) - (1.5) will reduce it to restrictive Taylor approximation.)

2. Applications of Approximations for IBVP for PDE

We begin with presenting some details and introduce through the advanced proofs that the used numerical solutions are convergent, from analytical point of view, to the exact solutions.

2.1 Convergence to the exact solution:

In Butcher (2008) proved that LTE $T_{ij/RP}$ for the restrictive Pade` approximation for parabolic type takes the form:

$$\begin{aligned}
&\left[3(u_t) + O(k,h^2)\right] T_{ij/RP}(h,k) = \\
&-\left\{ \frac{1}{8}\left[(u_{t^2})_{ij}(u_t)_{i0} - (u_{t^2})_{i0}(u_t)_{ij} \right] h^2 \right. \\
&+ \frac{1}{4}\left[(u_{t^3})_{ij}(u_t)_{i0} - (u_{t^3})_{i0}(u_t)_{ij} \right] k^2 \\
&+ \frac{1}{8}\left[(u_{t^4})_{ij}(u_t)_{i0} - (u_{t^4})_{i0}(u_t)_{ij} + (u_{t^3})_{ij}(u_{t^2})_{i0} - (u_{t^3})_{i0}(u_{t^2})_{ij} \right] k^3 \\
&\left. \qquad + \frac{1}{8}\left[(u_{t^3})_{ij}(u_t)_{i0} - (u_{t^3})_{i0}(u_t)_{ij} \right] k\, h^2 + \cdots \right\}.
\end{aligned}$$

The terms between each square brackets [ ] will take the form:

$$A_{p,m} = \left[(u_{t^p})_{ij}(u_{t^m})_{i0} - (u_{t^p})_{i0}(u_{t^m})_{ij} \right],\ p, m = 0(1)\infty$$

Using the closed form exact solution: $u(x,t)=e^{-t}\sin x$ where $x=ih,\ t=jk$

$A_{p,m} = (-1)^{p+m}\left[\left(e^{-t}\sin x\right)_{ij}\left(e^{-t}\sin x\right)_{i0} - \left(e^{-t}\sin x\right)_{i0}\left(e^{-t}\sin x\right)_{ij}\right] = 0$ i.e it can be shown that: $T_{ij/RP}(h,k) = 0$.

It is proved also that LTE $T_{ij/RP}$ for the restrictive Pade` approximation for hyperbolic type takes the form

$$\left[(u_t)_{i0} + O(k)\right]T_{ij/RP}(h,k) = \left\{\frac{a}{6}\left[(u_t)_{i0}\left(u_{x^3}\right)_{ij} - (u_t)_{ij}\left(u_{x^3}\right)_{i0}\right]h^2 + \frac{a}{24}\left[(u_t)_{i0}\left(u_{x^4}\right)_{ij} - (u_t)_{ij}\left(u_{x^4}\right)_{i0}\right]h^3 + \frac{a}{48}\left[\left(u_{t^2}\right)_{i0}\left(u_{x^3}\right)_{ij} - \left(u_{t^2}\right)_{ij}\left(u_{x^3}\right)_{i0}\right]h^2k + \cdots\right\}$$

The terms between each square brackets [ ] will take the form:

$$A_{p,m} = \left[\left(u_{t^p}\right)_{i0}\left(u_{x^m}\right)_{ij} - \left(u_{t^p}\right)_{ij}\left(u_{x^m}\right)_{i0}\right]$$

The advantages of the restrictive Pade` approximation can be summarized as follows:

i) The method gives the exact solution if it is known at one level of time, for example at $t= k$, *i.e.* $u(x,t)=u(ih,k)$ $i=1(1)N$.

ii) Without knowing the exact solution at one level, they tried to use an approximate, fast efficient and accurate method with suitable very small step sizes h and k, to get the needed almost exact solution at specific level, after which the usual restrictive Pade` process is continued.

iii) The needed exact solution at the first level need not be in closed form, i.e., we need only a table of the exact solution at some points.

In **(**Cohn, D, 2013) the authors obtained similar forms-but simpler- to the above forms. in the study of Taylor's approximations

In (Thomas, George B., Jr.; Finney, Ross L. 1996)**,** the authors studied the convergence of the restrictive Pade` approximation to the exact solution of nonlinear IBVP for generalized Burger's-Huxley and generalized Burger's-Fisher equations.

The corresponding LTE for Restrictive Pade method takes the form:

$$T_{i,j/RP} = \frac{1}{k}\left[-\frac{1}{2}r\left(1+\frac{1}{2}h\alpha M^{\delta}\right)u_{i-1} + \left(1+r+r\varepsilon_{i,l}+\frac{1}{2}rh^2\lambda\right)u_i - \frac{1}{2}r\left(1-\frac{1}{2}h\alpha M^{\delta}\right)u_{i+1}\right]_{j+1}$$
$$-\frac{1}{k}\left[-\frac{1}{2}r\left(1+\frac{1}{2}h\alpha M^{\delta}\right)u_{i-1} + \left(1-r+r\varepsilon_{i,l}+\frac{1}{2}rh^2\lambda\right)u_i - \frac{1}{2}r\left(1-\frac{1}{2}h\alpha u_{i,j}^{\delta}\right)u_{i+1}\right]_{j}$$
$$= \frac{1}{k}r\varepsilon_{i,l}\left(u_{i,j+1} - u_{i,j}\right) + T_{i,j/cn}$$

2.2. Restrictive Padé Approximation for Parabolic Types:

In this section we present the papers [20], [6], [5], and [11].

2.2.1 One Dimensional Case:

Restrictive Pade` Approximation for Parabolic PDEs

The derivation of the method is based on the simple heat–conduction standard problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 \le x \le x_0,\ t \ge 0 \qquad (.2.1)$$

with the initial and boundary conditions, respectively,

$$u(x,0) = g(x)\ 0 \le x \le x_0,\ u(0,t) = u(x_0,t) = 0,\ t \ge 0. \qquad (.2.2)$$

The open rectangular domain is covered by a rectangular grid with spacing h and k in the x, t directions respectively, the grid point (x, t) denoted by (ih, jk) and $u(ih, jk) = u_{i,i}$ where $i = 0(1)N$, j is a non –negative integer.

The exact solution of grid representation of (2.1) is given by

$$u_{i,j+1}=exp(kD^2_x)\, u_{i,j}\ ,$$

the approximation of the partial derivative D^2_x at the grid point (ih, jk) will take the usual form:

$$D^2_x u=\frac{1}{h^2}(u_{i-1,j}-2u_{i,j} + u_{i+1,j}) \qquad (2.3)$$

Then

$$U^{j+1} = exp(rA)\ U^j\ ,\ \ r = \frac{k}{h^2} \qquad (2.4)$$

where

$$U^j = (u_{1,j}\ ,u_{2,j}\ ,\ldots u_{N-1,j})^T,\ \ Nh = x_0$$

and

$$A = \begin{pmatrix} -2 & 1 & & & & 0 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -2 \end{pmatrix}_{(N-1)\times(N-1)} \qquad (2.5)$$

using the equation of the *RPA [0, 1, 1]* $_{exp(rA)}$ to approximate the exponential matrix in (2.4), then the vector form

$$U^{j+1} = \left(I + \left(\varepsilon - \frac{1}{2}A\right)r\right)^{-1}\left(I + \left(\varepsilon + \frac{1}{2}A\right)r\right)U^j = BU^j \qquad (2.6)$$

will have the stability condition: $-1/3 \le r\varepsilon_i \le 1,\ i = 1(1)N-1$.

The desired method is more efficient and more accurate than many known methods; also it is of low cost. The local truncation error in this method is less than that for classical known methods, in this method it varies from 10^{-10} to 10^{-14} *while* in the classical known methods it varies from 10^{-3} to10^{-7} as is done in the enclosed tables in (Ismail and Elbarbary, 2007). The efficiency of RP method is better than that for Douglas method because of the ratio of the number of multiplications and divisions between Douglas method and the constructed method is varies from 10:1 to 61:1, also this method is more fast than Douglas method because the ratio of the executive time of our desired method to Douglas method varies from 1:8 to 1:33.

2.2.2 Chebyshev Approach

Restrictive Chebyshev Rational Approximation and Applications to Heat-Conduction Problems

According to (Silviu-Ioan Filip, 2016), arestrictive type of Chebyshev rational approximation is constructed. It yields more accurate results and exact values at some selected points. Authors used this method to approximate the exponential function. This approach is applied to define a new implicit finite difference scheme to parabolic PDEs. This approach will exhibit several advantages features: highly accurate, fast, and the absolute error still very small whatever the exact solution is too large. Stability conditions are obtained and utilized in numerical computations. Finally, some numerical results to describe the performance of this approach are given.

In an approximation of a function, the property of the error that is of most importance is the maximum error (in magnitude) on the interval. Therefore, the major aim of the approximation process of a function is to make the maximum as small as possible. The rational approximations to a function lead to smaller maximum error than polynomial approximations. Maehly's method is an algorithm which enables one to convert Chebyshev series into a rational expressions involving Chebyshev polynomials. The Chebyshev expansion for the function f(x) is given by

$$f(x) = \frac{1}{2}c_0 + \sum_{j=1}^{\infty} c_i T_j(x),$$

where $c_j = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_j(x)}{\sqrt{1-x^2}}dx, j=0,1,\ldots$ and $T_j(x) = cos(jcos^{-1}x)$ is the j^{th} degree Chebyshev polynomial.

The Chebyshev rational approximation for the function f(x) will be

$$T_{m,n}(x) = \frac{\sum_{j=0}^{m} a_j T_j(x)}{\sum_{j=0}^{n} b_j T_j(x)},$$

where the a_j*'s* and b_j*'s* are to be determined, so that

$a_0 = \frac{1}{2}\sum_{i=0}^{n} b_i c_i$ and $a_r = \frac{1}{2}\sum b_i (c_{|r-i|} + c_{r+i})$, *r = 1, 2,…, (m+n),* $a_r = 0, \quad \forall r > m.$

In this paper authors developed a technique for generating a rational approximation to a function which give the exact values at certain selected points. Furthermore, they used the new approach to solve heat-conduction problems.

Restrictive Chebyshev rational approximation:

The restrictive Chebyshev rational approximation, of the function f(x) due to combining between rational approximation and interpolation. It can be written in the form:

$$T_{m,n}^{p}(x) = \frac{\sum_{i=0}^{m} a_i T_i(x) + \sum_{i=m+1}^{m+p} \varepsilon_{i-m} T_i(x)}{1 + \sum_{i=1}^{n} b_i T_i(x)},$$

where the a_i*'s* and b_i*'s* are to be determined so that

$$f(x) - T_{m,n}^{p}(x) = \frac{\sum_{i=0}^{\infty} c_i T_i(x)\left(1 + \sum_{j=1}^{n} b_i T_j(x)\right) - \sum_{i=0}^{m} a_i T_i(x) - \sum_{i=m+1}^{m+p} \varepsilon_{i-m} T_i(x)}{1 + \sum_{i=1}^{n} b_i T_i(x)}.$$

The coefficients of $T_i(x)$, $i=0,1,2,\ldots,(m+n)$ in the numerator of the right-hand side vanish. Hence

$$\sum_{i=0}^{\infty} c_i T_i(x)\left(1 + \sum_{j=1}^{n} b_j T_j(x)\right) - \sum_{i=0}^{m} a_i T_i(x) - \sum_{i=m+1}^{m+p} \varepsilon_{i-m} T_i(x) = \sum_{i=m+n+1}^{\infty} h_i T_i(x). \quad (2.7)$$

In order to get equations for the a_i*'s,* b_i*'s* and ε_i*'s* use the identity,

$$T_i(x) T_j(x) = \frac{1}{2}\left(T_{i+j}(x) + T_{|i=j|}(x)\right)$$

and with this, Eq. (1.2.7) can be rewritten as:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{n} c_i b_j \left(T_{i+j}(x) + T_{|i-j|}(x)\right) - \left(\sum_{i=0}^{m} a_i T_i(x) + \sum_{i=m+1}^{m+p} \varepsilon_{i-m} T_i(x)\right) = \sum_{i=m+n+1}^{\infty} h_i T_i(x), \quad (2.8)$$

where $b_0=1$. Then it is obtained

$$a_0 = \frac{1}{2}\sum_{i=0}^{n} b_i c_i, \; a_r = \frac{1}{2}\sum b_i (c_{r+i} + c_{|r-i|}), \qquad r=1,2,\ldots,m,$$

$$\varepsilon_{r-m} = \frac{1}{2}\sum b_i (c_{r+i} + c_{|r-i|}), \qquad r = m+1, m+2, \ldots, m+n, \quad \varepsilon_k = 0, \quad \forall k > l,$$

which are $(n + m + 1)$ equations in $(m + p + n + 1)$ unknowns, then putting the following interpolating equations

$$T_{m,n}^{p}(x_j) = f(x_j), \qquad j=1,2,\ldots,l,$$

which gives a system of l equations. It means that the suggested approximation is exact at l points. Then the $(m + n + p + 1)$ coefficients, a_i, $i=0,1,\ldots,m, b_i$, $i= 1,2,\ldots, m$ and $\varepsilon_{l,i}= 1,2,\ldots,l$ can be obtained from these systems. The points x_j, $j=1,2,\ldots,l$ are chosen such that the error $\left|f(x_j) - T_{m,n}(x_j)\right|$ is maximum.

Chebyshev rational approximation of the exponential matrix:

The approximation of the exponential matrix *exp(rH)* in terms of Cheby-shev polynomials can be defined as:

$$\exp(rH) = \frac{1}{2} c_0 I + \sum_{j=0}^{\infty} c_j T_j(rH),$$

Where $T_0(rH)=I$, $T_1(rH)=rH$, $T_{n+1}(rH)=2rHT_n(rH)-T_{n-1}(rH)$ and we can get c_j by the orthogonality property of Chebyshev approximation.

As an example, the Chebyshev rational approximation $T_{1,1}$ up to 15-digits is given by:

$$T_{1,1}(e^r)=\frac{a_0+a_1r}{1+b_1r}=\frac{1.004818014180699+0.4823222876616465r}{1-0.4622554267229283r}$$

where

$$b_1=-\frac{2c_2}{c_1+c_3},\ a_0=\frac{1}{2}(c_0+c_1b_1),\ a_1=c_1+\frac{1}{2}b_1(c_0+c_2),$$

and the restrictive Chebyshev rational approximation $T_{1,1}^1$ has the form

$$T_{1,1}^1(e^{rA})=(I+b_1rA)^{-1}(a_0I+a_1rA), \qquad (2.9)$$

where

$$b_1=-\frac{2(c_2-\varepsilon_1)}{c_1+c_3},\ a_0=\frac{1}{2}(c_0+c_1b_1),\ a_1=c_1+\frac{1}{2}b_1(c_0+c_2),$$

Restrictive Chebyshev rational approximation for parabolic PDE^s:

The derivation of the desired method is based on the simple heat-conduction standard problem (2.1): with initial and boundary conditions similar to (2.2).

The exact solution of a grid representation of Eq. (2.1) is given by

$$u_{i,j+1}=\exp\left(k\frac{\partial^2}{\partial x^2}\right)u_{i,j} \qquad (2.10)$$

where

$$\frac{\partial^2}{\partial x^2}=\frac{1}{h^2}\left(\delta_x^2-\frac{1}{12}\delta_x^4+\frac{1}{90}\delta_x^6+\cdots\right),\qquad r=\frac{k}{h^2}. \qquad (2.11)$$

If the difference formula, correct to second difference, is substituted for $\frac{\partial^2}{\partial x^2}$, then Eq. (2.11) can be rewritten as: $U_{i,j+1}=\exp(r\delta_x^2)U_{i,j}$

Then the following finite difference equation is obtained:

$b_1rU_{i-1,j+1}+(1-2b_1r)U_{i,j+1}+b_1rU_{i+1,j+1}=a_1rU_{i-1,j}+(a_0-2a_1r)U_{i,j}+a_1rU_{i+1,j}$ (2.12)

The stability condition takes the form

$$\frac{|a_0-2a_1r|+|2a_1r|}{|1-2b_1r|+|2b_1r|}\le 1.$$

2.2.3 Two Dimensional Case

According to, D. Ambrosi, D, Arioli, G. and Koch (2012), a new accurate fast implicit method for the finite difference solution of the two dimensional parabolic PDE with first level condition, which may be obtained by any other method. The stability region is discussed. The suggested method is considered as an accelerating technique for the implicit finite difference scheme, which is used to find the first level condition. The obtained results are compared with some famous finite difference schemes and it is satisfactory agreement with the exact solution.

Equations of motion in fluid mechanics are frequently reduced to parabolic formulations. In addition, the unsteady heat conduction equation is also parabolic. Also many of the applications parabolic equations arise as macroscopic descriptions or processes whose microscopic description is essentially probabilistic. Heat flow is related to the random motion of electrons and atoms. Parabolic equations are also used in economic models to give a macroeconomics description of market behavior.

The authors examined the suggested scheme, that can be used to solve a simple model of parabolic PDE^s, that is the heat equation, this equation is called the model equation because it can be used to model the behavior of more complicated parabolic PDE^s. The heat equation can serve as a model equation for other parabolic PDE^s such as the boundary layer equations. The model equation under consideration has the following form:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \qquad (2.13)$$

in the rectangular domain bounded by $0\le x\le x_0$ and $0\le y\le y_0$ with specific boundary and initial conditions.

The restrictive Pade` approximation is used to approximate the exponential form $\exp(r\delta_x^2)$ where δ_x^2 is the central difference operator.

For example the restrictive Pade` approximation *RPA[1,1]* or the exponential form $\exp(r\delta_x^2)$ is given by

$$RPA[1,1]_{\exp(\delta_x^2)}(r) = \left(1 + \left(\varepsilon_i - \frac{1}{2}\delta_x^2\right)r\right)^{-1}\left(1 + \left(\varepsilon_i + \frac{1}{2}\delta_x^2\right)r\right).$$

Various finite difference approximations can be used to represent the derivatives in Eq. (2.13), The resulting finite difference methods can be classified into two broad categories, explicit or implicit methods. The explicit methods has the advantages of being simple and easy to program, but the restriction of small time step usually requires a very large number of

time step. If we apply the simple explicit method to the two dimensional heat equation, the stability condition reduces to $r \le 0.25$ which is twice as restrictive as the one dimension constraint $r \le 0.5$ and makes this method even more impractical. To alleviate this difficulty various implicit methods have been devised. The resulting system of linear algebraic equations requires substantially more computer time to solve than do a tri-diagonal system. The difficulties described above, which occur attempting to solve the two dimensional heat equation by explicit or implicit algorithms, led to the development of alternating-direction implicit (ADI) methods.

In this paper authors introduced a new implicit method for solving two dimensional parabolic PDE^s, which will exhibit several advantageous features, such as higher accuracy compared with other known methods, the choice of the time step length is sufficiently large compared with that can be used for the classical schemes, this allows them to have the solution at high time level and the accuracy has not been lost when the value of the exact solution is sufficiently large. i.e., the absolute error is sufficiently small whenever the exact solution is relatively large. Their method is considered as acceleration technique for the classical finite difference methods.

Restrictive Pade` Approximation for 2-Dimensional Parabolic PDEs

The derivation of the method is based on the unsteady heat conduction Eqs. (2.13). The region to be examined in *(x,y,t)* space is covered by a rectilinear grid with sides parallel to axes with h and k the grid spacing in the distance and time directions respectively. The grid point *(x,y,t)* are given by *(ph, mh, nk)* where *p, m =0(l)N and n = 0, 1, 2,...* . The function satisfying the difference equation at the grid point is $u_{p,m}^n$.

The exact difference replacement of Eq, (1.2.13) is given by

$$u_{p,m}^{n+1} = \left(\exp\left(k\frac{\partial^2}{\partial x^2}\right)\exp\left(k\frac{\partial^2}{\partial y^2}\right)\right)u_{p,m}^n \qquad (2.14)$$

where $\partial^2/\partial x^2$ is as in (2.11) and so is $\partial^2/\partial y^2$.

If the difference formulae, correct to second difference are substituted for $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ Eq. (2.14) can be rewritten as:

$$u_{p,m}^{n+1} = \left(\exp\left(r\delta_x^2\right)\exp\left(r\delta_y^2\right)\right)u_{p,m}^n, \text{ where } r = \frac{k}{h^2} \qquad (2.15)$$

If we use Pade` approximation PA[1, 1] to approximate $\exp(r\delta_x^2)$ and $\exp(r\delta_y^2)$, then we get the Crank-Nicolson formula (1947):

$$(1-0.5r\delta_x^2)(1-0.5r\delta_y^2)u_{p,m}^{n+1} = (1+0.5r\delta_x^2)(1+0.5r\delta_y^2)u_{p,m}^n \quad (2.16)$$

Also Eq. (2.15) can be approximate in the form

$$u_{p,m}^{n+1} = \left(1 + k\frac{\delta^2}{\delta x^2}\right)\left(1 + k\frac{\delta^2}{\delta y^2}\right)u_{p,m}^n \qquad (2.17)$$

by using *PA[0, 1]* for the difference operator (*1 - (1/12)* δ_x^2 *+ (1/90)* δ_x^4 +…), $\partial^2/\partial x^2$ can be approximated in the form:

$$\frac{\partial^2}{\partial x^2} = \frac{1}{h^2}\frac{\delta_x^2}{1+(1/12)\delta_x^2},$$

similarly

$$\frac{\partial^2}{\partial y^2} = \frac{1}{h^2}\frac{\delta_y^2}{1+(1/12)\delta_y^2}.$$

Substituting in (1.2.17) and by expanding to given Douglas and Rachford formula:

$$\left(1-\frac{1}{2}\left(r-\frac{1}{6}\right)\delta_x^2\right)\left(1-\frac{1}{2}\left(r-\frac{1}{6}\right)\delta_y^2\right)u_{p,m}^{n+1} = \left(1+\frac{1}{2}\left(r+\frac{1}{6}\right)\delta_x^2\right)\left(1+\frac{1}{2}\left(r+\frac{1}{6}\right)\delta_y^2\right)u_{p,m}^{n} \qquad (2.18)$$

Both these methods in Eqs. (1.2.16) and (1.2.18) are unconditionally stable. However, the use of these methods requires the solution of a large number of simultaneous linear algebraic equations at each time step. Now using restrictive Pade\` approximation *RPA[1,1]* to approximate $\exp(r\delta_x^2)$, and $\exp(r\delta_y^2)$, then:

$$(1+\varepsilon_{p,m}r-0.5r\delta_x^2)(1+\varepsilon_{p,m}r-0.5r\delta_y^2)u_{p,m}^{n+1} = (1+\varepsilon_{p,m}r+0.5r\delta_x^2)(1+\varepsilon_{p,m}r+0.5r\delta_x^2)u_{p,m}^{n} \qquad (2.19)$$

where the constants $\varepsilon_{p,m}$ are to be determined. Then we must know an additional condition, *(x,y,k)* to be given, i.e., $\varepsilon_{p,m}$ must be given such that the truncation error at certain r is zero, after which, use (2.19) for another level for calculations. Then the method suggested by (2.19) will be stable when $r\varepsilon_{p,m} \geq -1$, *p, m =1(1)N.*

Conclusion

The suggested method is more efficient and more accurate than many known methods, also it is of low cost. The local truncation error in our method is less than that for the classical known methods, in our method it varies from *I0*$^{-}$*8* to 10^{-15} while in the classical known methods it varies from 10^{-2} *to*10^{-9}. The efficiency of our method is better than that for ***Douglass and Rachford (D.R.)*** method because of the ratio of the number of systems to be solved between D.R. method and this method varies from 2: 1 to 10: 1, also their method is more fast than D.R. method because the ratio of the executed time of their suggested method to D.R. method varies from 1: 2 to 1: 15 the suggested method is considered as an acceleration technique for the classical finite difference methods.

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