Additive processes with return to zero and Stoynov processes

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Abstract.
Two families of processes are considered in this article – additive processes with return to zero and Stoynov processes.
Additive processes with return to zero are a specifc kind of generalized additive processes.
Stoynov distributions and processes or also Switch Time distributions and processes (ST distributions and processes) can be considered as specific kind of generalized gamma distributions and their corresponding processes.
Here by switch time we denote the time of sharp change of value of a stochastic process (jump) or sharp change of some characteristics of a stochastic process (regime switch).

Key words: processes with return to zero, Switch-time family distributions and processes, Stoynov family distributions and processes

Introduction. Additive processes – characterization and possible generalisations

With a given filtered probability space (Ω, F, F_t, P) satisfying the usual conditions, an (one or d-dimensional) additive process is defined as a stochastic process $\{X(t), t \geq 0\}$ which is càdlàg and satisfies the following conditions:

1. $X(0) = 0;$
2. The process has independent increments, i. e. for every sequence of numbers

$$0 < t_1 < t_2 < \cdots < t_n$$

the random variables

$$X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent;

3. The process is stochastically continuous (continuous in probability), i. e.

$$\lim_{t \to s} X(t) = X(s),$$

where the limit is taken by probability.

The additive processes are introduced by Lévy (1937, 1992). Their properties are studied in Sato (1999). In Sato (1999), the following theorem is presented:

Theorem (Sato) Let $\{X(t), t \geq 0\}$ be an additive process with values in R^d. Then:

1. For every $t \geq 0$ the random variable $X(t)$ has infinitely divisible distribution.
2. The distribution of $\{X(t), t \geq 0\}$ is uniquely identified by its point characteristics $\{(A(t), \mu(t), \Gamma(t)), t \geq 0\}$, which are related to the characteristic function in the following way:

$$Ee^{iuX(t)} = e^{\psi(t,u)}$$

and

$$\psi(t,u) = -\frac{1}{2} uA(t)u + iu\Gamma(t) + \int_{R^d} \mu(t,dx)(e^{iux} - 1 - iux1_{\{|x| \leq 1\}}).$$

The point characteristics of the additive process satisfy the properties:

2.1. For every $t \geq 0$ $A(t)$ is positive definite matrix with dimension $d \times d$, $\Gamma(t)$ is a function with values which are vectors in R^d and $\mu(t,dx)$ is a positive measure in R^d, satisfying the conditions $\mu(t,\{0\}) = 0$ and $\int_{R^d} \min(|x|^2, 1)\mu(t,dx) < \infty.$

2.2. $A(0)=0, \mu(0,dx)=0, \Gamma(0)=0$ and for all $t \geq s \geq 0$ the matrix $A(t)-A(s)$ is a positive definite matrix with dimension $d \times d$ and $\mu(s,B) \leq \mu(t,B)$ for all measurable sets $B \in \mathrm{B}(R^d)$.

2.3. If $s \to t$, then $A(s) \to A(t), \Gamma(s) \to \Gamma(t), \mu(s,B) \to \mu(t,B)$ for all measurable sets $B \in \mathrm{B}(R^d)$, for which $B \subset \{x: |x| \geq \varepsilon\}$ for some $\varepsilon > 0$.

Conversely, for the family triples $\{(A(t), \mu(t), \Gamma(t)), t \geq 0\}$, which satisfies 2.1., 2.2. and 2.3. there is an additive process $\{X(t), t \geq 0\}$ with the corresponding point characteristics.

Important special cases of the additive processes are the processes for which there exists the following parametric representation of the point characteristics:

1. $A(t) = \int_0^t \sigma^2(s)ds$, where for every $t \geq 0$ the matrix $\sigma(t)$ is a real matrix with dimension $d \times n$ for which $\int_0^T \sigma^2(t)dt < \infty$ for fixed $T > 0$ such that $\sigma(t)$ is defined in the interval $[0,T]$.

2. $\mu(t,B) = \int_0^t \nu(s,B)ds$ for all measurable sets $B \in \mathrm{B}(R^d)$, where $\{\nu(t), t \in [0,T]\}$ is a family of Lévy measures satisfying the condition $\int_0^T \int_{R^d} \min(|x|^2,1)\nu(t,dx)dt < \infty$.

3. $\Gamma(t) = \int_0^t \gamma(s)ds$, where $\gamma:[0,T] \to R$ is a deterministic function with finite variation.

For the triple $(\sigma^2(t), \nu(t), \gamma(t))$ we say that it defines the local characteristics of the additive process. The additive processes with local characteristics are semimartingales.

There are different possibilities for generalization of the additive processes.

One way of generalization is to introduce jumps in deterministic moments, as proposed by Kallsen (1998). In this way, the process is not already stochastically continuous and with stationary increments. To characterize such kind of processes, to the point and local characteristics there is a need to add new characteristics identifying the position and the size of these deterministic-time jumps.

The point characteristics become

$$\{(\Theta, K(t), A(t), \mu(t), \Gamma(t)), t \geq 0\},$$

where Θ is a discrete set giving the times of the jumps and $K(t)$ is conditional distribution of the jumps for which

$$K(t,G) = \begin{cases} \mu(t,G) + \varepsilon_0(G)(1-\mu(t,R)), t \in \Theta \\ 0, t \notin \Theta. \end{cases}$$

Here $\varepsilon_0(G)$ is the Dirac measure in the point 0.

In the case when local characteristics exist we have

$$\{(\Theta, K(t), A(t) = \int_0^t \sigma^2(s)ds,$$

$$\mu(t,G) = \int_0^t \nu(s,G)ds + \sum_{s \in \Theta \cap [0,t]} \mu(s,G) = \int_0^t \nu(s,G)ds + \sum_{s \in \Theta \cap [0,t]} K(s, G \setminus \{0\}),$$

$$\Gamma(t) = \int_0^t \gamma(s)ds + \sum_{s \in \Theta \cap [0,t]} \Delta\Gamma(s) = \int_0^t \gamma(s)ds + \sum_{s \in \Theta \cap [0,t]} \int xK(s,dx)), t \geq 0\},$$

and the local characteristics are:

$$\{(\Theta, K(t), \sigma^2(t), \nu(t), \gamma(t)), t \geq 0\}.$$

A next step of generalization is to allow stochastic characteristics- drift, volatility and jump intensity. The corresponding processes are considered for example in Grigelionis (1973).

In this case, the point characteristics of the process are

$$\{(\Theta, K(\omega,t), A(\omega,t), \mu(\omega,t), \Gamma(\omega,t)), t \geq 0\}$$

and the local characteristics (when they exist) are

$$\{(\Theta, K(\omega,t), \sigma^2(\omega,t), \nu(\omega,t), \gamma(\omega,t)), t \geq 0\}.$$

The obtained characterization allows to specify also some specific jumps in random times and to characterize them separately from the remaining jumps of the process. In this case, the point characteristics can be presented as:

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t), \mu(\omega,t), \Gamma(\omega,t)), t \geq 0\}$$

where the random times of the specific jumps are represented by the point process $\{\Theta(\omega,t), t \geq 0\}.$

It is also possible to combine specific jumps in specific random times with jumps in deterministic moments. In this case, the parameters of the process (in the case when local characteristics exist) are:

$$\{(\Theta_1, \Theta_2(\omega,t), K_1(\omega,t), K_2(\omega,t), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds,$$

$$\mu(\omega,t,G) = \int_0^t \nu(\omega,s,G)ds + \sum_{s\in\Theta_1\cap[0,t]} K_1(\omega,s,G\setminus\{0\}) + \sum_{\theta_i<t} K_2(\omega,\theta_i,G\setminus\{0\}),$$

$$\Gamma(t) = \int_0^t \gamma(s)ds + \sum_{s\in\Theta_1\cap[0,t]} \int xK_1(\omega,s,dx) + \sum_{\theta_i<t} \int xK_2(\omega,\theta_i,dx)), t \geq 0\}.$$

Here $K_1(\omega,t)$ and $K_2(\omega,t)$ are the conditional distributions of the jumps in deterministic and specific random times correspondingly.

Additive processes with return to zero of type „jump” in deterministic moments

We consider processes $X(t) = G(t) + J(t),$ where $G(t)$ as an additive process, and $J(t)$ is a process of jumps which ensure the returns to zero.

Their point characteristics are given by

$$\{(\Theta, K(t,B), A(t), \mu(t,B) = \mu_A(t,B) + \sum_{s\in\Theta\cap[0,t]} K(s,B\setminus\{0\}),$$

$$\Gamma(t) = \Gamma_A(t) + \sum_{s\in\Theta\cap[0,t]} \int xK(s,dx)), t \geq 0\}.$$

The set Θ is a discrete set giving the moments of jumps. $K(t,B)$ is the conditional distribution of the jumps, for which we have

$$K(t,B) = \begin{cases} \mu(t,B) + \varepsilon_0(B)(1-\mu(t,R)), t \in \Theta \\ 0, t \notin \Theta. \end{cases}$$

Here $\varepsilon_0(B)$ is the Dirak measure in zero and $\mu_A(t,B)$ and $\Gamma_A(t)$ are the corresponding characteristics of the additive process $G(t).$

The return into zero is a constraint for the process $J(t)$ and leads to additional characterization of $K(t,B).$ So, we have the following lemma:

Lemma 1. For the additive processes with return to zero of type „jump” in deterministic moments, given by the set $\Theta,$ the conditional distributions of the jmps $K(t,B)$ is given by:

$$K(t,B) = \begin{cases} \varepsilon_{-X(t-)}(B), t \in \Theta \\ 0, t \notin \Theta. \end{cases}$$

Proof: We have $K(t,B) = P_{\Delta X(t)|F_{t-}}(B),$ i. e. $K(t)$ is the conditional distribution of the jumps. For the processes with returns to zero we have:

$$\Delta X = X(t) - X(t-) = \begin{cases} 0 - X(t-) = -X(t-) = J(t), t \in \Theta \\ G(t) - G(t-), t \notin \Theta. \end{cases}$$

□

Corrolary 1. At the points of Θ for processes with returns to zero we have

$$\sum_{s \in \Theta \cap [0,t]} K(s, B \setminus \{0\}) = \sum_{s \in \Theta \cap [0,t]} \varepsilon_{-X(s-)}(B \setminus \{0\}).$$

Proof: It foolows from Lemma 1 applied to the left part of the equation.. □

Corrolary 2. We have:

$$\mu(t, B) = \mu_A(t, B) + \sum_{s \in \Theta \cap [0,t]} \varepsilon_{-X(s-)}(s, B \setminus \{0\}).$$

Proof: It follows from

$$\mu(t, B) = \mu_A(t, B) + \sum_{s \in \Theta \cap [0,t]} K(s, B \setminus \{0\}) =$$

$$= \mu_A(t, B) + \sum_{s \in \Theta \cap [0,t]} \varepsilon_{-X(s-)}(s, B \setminus \{0\}).$$ □

Corrolary 3. We have the equation

$$\int xK(s, dx) = J(s).$$

Proof: We have

$$\int xK(s, dx) = \int x\varepsilon_{-X(s-)}(dx) = -X(s-) = J(s).$$ □

Corrolary 4. We have the equation:

$$\Gamma(t) = \Gamma_A(t) - \sum_{s \in \Theta \cap [0,t]} X(s-).$$

Proof: We have

$$\Gamma(t) = \Gamma_A(t) + \sum_{s \in \Theta \cap [0,t]} \Delta\Gamma(s) =$$

$$= \Gamma_A(t) + \sum_{s \in \Theta \cap [0,t]} \int xK(s, dx) =$$ □

$$= \Gamma_A(t) - \sum_{s \in \Theta \cap [0,t]} X(s-).$$

Corrolary 5. In the case where the additive processes with returns to zero of type "jump" are with local characteristics, the point characteristics are:

$$\{(\Theta, K(t, B), A(t) = \int_0^t \sigma^2(s)ds, \mu(t, B) =$$

$$= \int_0^t \nu(s, B)ds + \sum_{s \in \Theta \cap [0,t]} \varepsilon_{-X(s-)}(B \setminus \{0\}),$$

$$\Gamma(t) = \int_0^t \gamma(s)ds - \sum_{s \in \Theta \cap [0,t]} X(s-)), t \geq 0\}$$

and the local characteristics are

$$\{(\Theta, K(t, B), \sigma^2(t), \nu(t), \gamma(t)), t \geq 0\}.$$

Proof: We have

$$\{(\Theta, K(t,B), A(t) = \int_0^t \sigma^2(s)ds, \mu(t,B) =$$

$$= \int_0^t \nu(s,B)ds + \sum_{s\in\Theta\cap[0,t]} \mu(s,B) =$$

$$= \int_0^t \nu(s,B)ds + \sum_{s\in\Theta\cap[0,t]} K(s,B\setminus\{0\}) =$$

$$= \int_0^t \nu(s,B)ds + \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(B\setminus\{0\}),$$

$$\Gamma(t) = \int_0^t \gamma(s)ds + \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(s) = \int_0^t \gamma(s)ds - \sum_{s\in\Theta\cap[0,t]} X(s-)), t \geq 0\}.$$

The local characteristics are determined from here immediately. □

Corrolary 6. In the case of Levy process with returns to zero of type "jump" in deterministic moments the point characteristics are

$$\{(\Theta, K(t,B), A(t) = \sigma^2 t, \mu(t,B) = t\nu + \sum_{s\in\Theta\cap[0,t]} \mu(s,B) =$$

$$= t\nu + \sum_{s\in\Theta\cap[0,t]} K(s,B\setminus\{0\}) = t\nu + \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(B\setminus\{0\}),$$

$$\Gamma(t) = \gamma t + \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(s) = \gamma t - \sum_{s\in\Theta\cap[0,t]} X(s-)), t \geq 0\}$$

and the local characteristics are

$$\{(\Theta, K(t,B), \sigma^2, \nu, \gamma), t \geq 0\}.$$

Proof: It follows from Corrolary 5 and from the from the characteristics of the Levy processes. □

Corrolary 7. For Brownian motion without drift with returns to zero of type "jump" in deterministics moments the point characteristics are

$$\{(\Theta, K(t,B), A(t) = \sigma^2 t, \mu(t,B) = \sum_{s\in\Theta\cap[0,t]} \mu(s,B) =$$

$$= \sum_{s\in\Theta\cap[0,t]} K(s,B\setminus\{0\}) = \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(B\setminus\{0\}),$$

$$\Gamma(t) = \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(s) = \sum_{s\in\Theta\cap[0,t]} \int xK(s,dx) =$$

$$= -\sum_{s\in\Theta\cap[0,t]} X(s-)), t \geq 0\},$$

and the corresponding local characteristics are:

$$\{(\Theta, K(t,B), \sigma^2, 0, 0), t \geq 0\}.$$

Proof: It foolos from Corrolary 5 and from the characteristics of the Brownian motion. □

Corrolary 8. For Poisson process with returns to zero of type "jumps" in deterministic moments the point (integral) characteristics are

$$\{(\Theta, K(t,B), A(t) = 0,$$

$$\mu(t,B) = \lambda ft + \sum_{s\in\Theta\cap[0,t]} \mu(s,B) = \lambda ft + \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(B\setminus\{0\}),$$

$$\Gamma(t) = \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(s) = -\sum_{s\in\Theta\cap[0,t]} X(s-)), t \geq 0\},$$

and the corresponding local characteristics are

$$\{(\Theta, K(t,B), 0, f\lambda, 0), t \geq 0\}.$$

Proof: It follows from Corrolary 5 and from the characteristics of the Poisson process. □

Additive processes with random characteristics and returns to zero of type “jump” in deterministic moments

They are processes $X(t) = G(t) + J(t),$ where $G(t)$ is an additive process and $J(t)$ is the jump process ensuring the returns to zero having random (stochastic) characteristics for $G(t)$ and $J(t).$

The additive processes with stochastic characteristics and returns to zero of type “jump” in deterministic moments have the following integral characteristics:

$$\{(\Theta, K(\omega,t,B), A(\omega,t), \mu(\omega,t,B) =$$
$$= \mu_A(\omega,t,B) + \sum_{s\in\Theta\cap[0,t]} \mu(\omega,s,B),$$
$$\Gamma(\omega,t) = \Gamma_A(\omega,t) + \sum_{s\in\Theta\cap[0,t]} \int xK(\omega,s,dx)), t \geq 0\}.$$

The returns to zero are constraints for the process $J(t)$ and lead to additional charactiristization of $K(\omega,t,B),$ given by the following lemma:

Lemma 2. For the additive process with stochastic characteristics and returns to zero of type “jump” in deterministic moments given by the set $\Theta,$ the conditional distribution of the jumps $K(\omega,t,B)$ is:

$$K(\omega,t,B) = \begin{cases} \varepsilon_{-X(t-)}(\omega,B), t \in \Theta \\ 0, t \notin \Theta. \end{cases}$$

Proof: Similar to the proof of Lemma1. □

Corrolary 2.9. When for the additive processes with stochastic characteristics and retrns to zero of type “jump” in deterministic moments there exist also local characteristics, the integral characteristics are

$$\{(\Theta, K(\omega,t,B), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds, \mu(\omega,t,B) =$$
$$= \int_0^t \nu(\omega,s,B)ds + \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(B \setminus \{0\}),$$
$$\Gamma(\omega,t) = \int_0^t \gamma(\omega,s)ds - \sum_{s\in\Theta\cap[0,t]} X(s-)), t \geq 0\}.$$

and the local characteristics are:

$$\{(\Theta, K(\omega,t,B), \sigma^2(\omega,t), \nu(\omega,t,B), \gamma(\omega,t)), t \geq 0\}.$$

Proof: We have

$$\{(\Theta, K(\omega,t,B), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds, \mu(\omega,t,B) =$$
$$= \mu_A(\omega,t,B) + \sum_{s\in\Theta\cap[0,t]} \mu(\omega,s,B) =$$
$$= \int_0^t \nu(\omega,s,B)ds + \sum_{s\in\Theta\cap[0,t]} K(\omega,s,B \setminus \{0\}) =$$
$$= \int_0^t \nu(\omega,s,B)ds + \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(B \setminus \{0\}),$$
$$\Gamma(\omega,t) = \int_0^t \gamma(\omega,s)ds + \sum_{s\in\Theta\cap[0,t]} \int xK(\omega,s,dx) =$$
$$= \int_0^t \gamma(\omega,s)ds + \sum_{s\in\Theta\cap[0,t]} X(s-)), t \geq 0\}.$$

From here and from local characteristics we see the result immediately. □

Corrolary 10. In the case of Levy processes with stochastic characteristics and returns to zero of type "jump" in deterministic moments the integral characteristics are

$$\{(\Theta, K(\omega,t), A(\omega,t) = t\sigma^2(\omega), \mu(\omega,t,B) =$$
$$= t\nu(\omega,B) + \sum_{s\in\Theta\cap[0,t)} K(\omega,s,B\setminus\{0\}) =$$
$$= t\nu(\omega,B) + \sum_{s\in\Theta\cap[0,t)} \varepsilon_{-X(s-)}(B\setminus\{0\}),$$
$$\Gamma(\omega,t) = t\gamma(\omega) + \sum_{s\in\Theta\cap[0,t)} \int xK(\omega,s,dx) =$$
$$= t\gamma(\omega) - \sum_{s\in\Theta\cap[0,t)} X(s-)), t\geq 0\}$$

and the local characteristics are

$$\{(\Theta, K(\omega,t), \sigma^2(\omega,t) = \sigma^2(\omega), \nu(\omega,t,B)$$
$$= \nu(\omega,B), \gamma(\omega,t) = \gamma(\omega)), t\geq 0\}.$$

Proof: We use Corrolary 9 and the characteristics of the Levy processes. □

Corrolary 11. For the Brownian motion with stochastic characteristics nad returns to zero of type "jump" in deterministic moments the point characteristics are

$$\{(\Theta, K(\omega,t), A(\omega,t) = t\sigma^2(\omega) = t\sigma^2(\omega),$$
$$\mu(\omega,t,B) = \sum_{s\in\Theta\cap[0,t)} K(\omega,s,B\setminus\{0\}) = \sum_{s\in\Theta\cap[0,t)} \varepsilon_{-X(s-)}(B\setminus\{0\}),$$
$$\Gamma(\omega,t) = \sum_{s\in\Theta\cap[0,t)} \int xK(\omega,s,dx) = -\sum_{s\in\Theta\cap[0,t)} X(s-)), t\geq 0\}.$$

and the local characteristics are

$$\{(\Theta, K(\omega,t), A(\omega) = \sigma^2(\omega), \nu(\omega,B) = 0,$$
$$\gamma(\omega,t) = 0), t\geq 0\}.$$

Proof: It follows from Corrolary 9 and from the characteristics of the Brownian motion. □

Corrolary 12. For the Poisson process with returns to zero of type "jump" in deterministic moments and stochastic characteristics the point characteristics are

$$\{(\Theta, K(\omega,t), A(\omega,t) = 0,$$
$$\mu(\omega,t) = ft\lambda(\omega) + \sum_{s\in\Theta\cap[0,t)} K(\omega,s,B\setminus\{0\}) =$$
$$= ft\lambda(\omega) + \sum_{s\in\Theta\cap[0,t)} \varepsilon_{-X(s-)}(B\setminus\{0\}),$$
$$\Gamma(\omega,t) = \sum_{s\in\Theta\cap[0,t)} \int xK(\omega,t,dx) = -\sum_{s\in\Theta\cap[0,t)} X(s-)), t\geq 0\}$$

and the local characteristics are:

$$\{(\Theta, K(\omega,t), 0, \nu(\omega) = \lambda(\omega)f, 0), t\geq 0\}.$$

Proof: We use Corrolary 9 and the characteristics of the Poisson process. □

Additive processes with ctochastic characteristics and returns to zero of type "jump" in specific random times

At the additive processes with stochastic characteristics and returns to zero of type "jump" at specific random moments among the jumps of the processes we choose specific jumps which happenat moments defined by the point process $\{\Theta(\omega,t), t\geq 0\}$.

Here again the retirn to zero is a constraint for the process $J(t)$ and it gives additional characterization of $K(\omega,t)$. More precisely, we have the following lemma.

Lemma 3. For the additive processes with stochastic characteristics and returns to zero of type "jump" at specific random moments given by the point process $\{\Theta(\omega,t), t \geq 0\}$, the conditional distribution of the jumps is

$$K(\omega,t,B) = \begin{cases} \varepsilon_{-X(t-)}(\omega,B), t \in \Theta \\ 0, t \notin \Theta. \end{cases}$$

Proof: Similar to the proofs of Lemma 1 and Lemma 2. □

Corrolary 13. The additive processes with stochastic characteristics and returns to zero of type "jump" at specific random times have the following integral characteristics:

$$\begin{aligned} &\{(\Theta(\omega,t), K(\omega,t), A(\omega,t), \mu(\omega,t,B) = \\ &= \mu_A(\omega,t,B) + \sum_{\theta_i<t} \varepsilon_{-X(\theta_i-)}(B \setminus \{0\}), \Gamma(\omega,t) = \\ &= \Gamma_A(\omega,t) - \sum_{\theta_i<t} X(\theta_i-)), t \geq 0\}, \end{aligned}$$

where θ_i are the moments of jumps of the process $\{\Theta(\omega,t), t \geq 0\}$.

Proof: The result of Lemma 3 is directly replaced in the general characteristics of the process:

$$\begin{aligned} &\{(\Theta(\omega,t), K(\omega,t,B), A(\omega,t), \mu(\omega,t,B) = \\ &= \mu_A(\omega,t,B) + \sum_{\theta_i<t} K(\omega,\theta_i, B \setminus \{0\}), \\ &\Gamma(\omega,t) = \Gamma_A(\omega,t) + \sum_{\theta_i<t} \int xK(\omega,s,dx). \end{aligned}$$

We obtain

$$\begin{aligned} (2.48)\quad &\{(\Theta(\omega,t), K(\omega,t), A(\omega,t), \mu(\omega,t,B) = \\ &= \mu_A(\omega,t,B) + \sum_{\theta_i<t} K(\omega,\theta_i, B \setminus \{0\}) = \\ &= \mu_A(\omega,t,B) + \sum_{\theta_i<t} \varepsilon_{-X(\theta_i-)}(B \setminus \{0\}), \Gamma(\omega,t) = \\ &= \Gamma_A(\omega,t) + \sum_{\theta_i<t} \int xK(\omega,s,dx) = \\ &= \Gamma_A(\omega,t) - \sum_{\theta_i<t} X(\theta_i-)), t \geq 0\}. \end{aligned}$$

□

Corrolary 14. In the case where for the additive processes with stochastic characteristics and returns to zero of type "jump" at specific random moments there exist also local characteristics, the point characteristics can b epresented as:

$$\begin{aligned} &\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds, \\ &\mu(\omega,t,B) = \int_0^t \nu(\omega,s,B)ds + \sum_{\theta_i<t} \varepsilon_{-X(\theta_i-)}(B \setminus \{0\}), \\ &\Gamma(\omega,t) = \int_0^t \gamma(\omega,s)ds - \sum_{\theta_i<t} X(\theta_i -)), t \geq 0\}. \end{aligned}$$

and the local characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), \sigma^2(\omega,t), \nu(\omega,t,B), \gamma(\omega,t)), t \geq 0\}.$$

Proof: We have

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds,$$

$$\mu(\omega,t,B) = \int_0^t \nu(\omega,s,B)ds + \sum_{\theta_i<t} K(\omega,\theta_i, B\setminus\{0\}) =$$

$$= \int_0^t \nu(\omega,s,B)ds + \sum_{\theta_i<t} \varepsilon_{-X(\theta_i-)}(B\setminus\{0\}),$$

$$\Gamma(\omega,t) = \int_0^t \gamma(\omega,s)ds + \sum_{\theta_i<t} \Delta\Gamma(\omega,\theta_i) =$$

$$= \int_0^t \gamma(\omega,s)ds - \sum_{\theta_i<t} X(\theta_i -)), t \geq 0\}.$$

From here the local characteristics are determined immediately. □

Corrolary 15. In the case of Levy processes with stochastic characteristics and returns to zero of type "jump" in random moments the point characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) = t\sigma^2(\omega),$$

$$\mu(\omega,t,B) = t\nu(\omega,B) + \sum_{\theta_i<t} K(\omega,\theta_i,B) =$$

$$= t\nu(\omega,B) + \sum_{\theta_i<t} \varepsilon_{-X(\theta_i-)}(B\setminus\{0\}),$$

$$\Gamma(\omega,t) = t\gamma(\omega) + \sum_{\theta_i<t} \Delta\Gamma(\omega,\theta_i) =$$

$$= t\gamma(\omega,t) + \sum_{\theta_i<t} \int xK(\omega,\theta_i,dx) =$$

$$= t\gamma(\omega,t) - \sum_{\theta_i<t} X(\theta_i -)), t \geq 0\}$$

and the local characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), \sigma^2(\omega), \nu(\omega,B), \gamma(\omega)), t \geq 0\}.$$

Proof: We use Corrolary 14 and the characteristics of the Levy processes. □

Corrolary 16. In the case of Brownian motion with stochastic characteristics and returns to zero of type "jump" at random moments the point characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) = t\sigma^2(\omega),$$

$$\mu(\omega,t,B) = \sum_{\theta_i<t} K(\omega,\theta_i,B\setminus\{0\}) =$$

$$= \sum_{\theta_i<t} \varepsilon_{-X(\theta_i-)}(B\setminus\{0\}),$$

$$\Gamma(\omega,t) = \sum_{\theta_i<t} \Delta\Gamma(\omega,\theta_i) = \sum_{\theta_i<t} \int xK(\omega,\theta_i,dx) =$$

$$= -\sum_{\theta_i<t} \int X(\theta_i -)), t \geq 0\}$$

and the local characteristics are

$$\{(\Theta(\omega,t), K(\omega,t,B), \sigma^2(\omega), \nu(\omega,B) = 0, \gamma(\omega) = 0), t \geq 0\}.$$

Proof: We use 9 and the characteristics of the Brownian motion. □

Corrolary 17. In the case of Poisson process with stochastic characteristics and returns to zero of type "jump" at random moments the point characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t)=0,$$
$$\mu(\omega,t,B)=t\lambda(\omega)f(B)+\sum_{\theta_i<t}K(\omega,\theta_i,B\setminus\{0\})=$$
$$=t\lambda(\omega)f(B)+\sum_{\theta_i<t}\varepsilon_{-X(\theta_i)}(B\setminus\{0\}),$$
$$\Gamma(\omega,t)=\sum_{\theta_i<t}\int xK(\omega,\theta_i,dx)=-\sum_{\theta_i<t}X(\theta_i-)),t\geq 0\}$$

and the local characteristics are

$$\{(\Theta(\omega,t), K(\omega,t,B), A(\omega)=0, \nu(\omega,B)=$$
$$=\lambda(\omega)f(B), \gamma(\omega)=0), t\geq 0\}.$$

Proof: We use Corrolary 9 and the characteristics of the Poisson process.. □

Additive processes with returns to zero of type „termination” in deterministic moments

These are additive by parts processes $X(t)=\sum_i G_i(t-t_i)1_{[t_i,t_{i+1})}(t),$ where $G_i(t)$ are additive processes, t_i are moments of returns to zero, and $1_A(t)$ is the indicator of the event A.

The additive processes with return to zero of type „termination” at deterministic moments have the following characteristics:

$$\{(\Theta=\{t_i\}, K(t), A(t)=\sum_i A_i(t-t_i)1_{[t_i,t_{i+1})}(t),$$
$$\mu(t,B)=\sum_i \mu_i(t-t_i,B)1_{[t_i,t_{i+1}))}(t)+\sum_{s\in\Theta\cap[0,t]}K(s,B\setminus\{0\}),$$
$$\Gamma(t)=\sum_i \Gamma_i(t-t_i)1_{[t_i,t_{i+1}))}(t)+\sum_{s\in\Theta\cap[0,t]}\int xK(s,dx)), t\geq 0\}.$$

Here $A_i(t)$, $\mu_i(t,B)$ and $\Gamma_i(t)$ are the corresponding characteristics of the additive processes $G_i(t)$. The set Θ is a discrete set giving the times of jumps. $K(t)$ is the conditional distribution of the jumps.

The return to zero is related to additional characterization of $K(t)$. The characterization may be obtained similar to the previous cases.

When the additive processes with returns to zero of type „termination” are with local characteristics, the point characteristics are

$$\{(\Theta=\{t_i\}, K(t,G), A(t)=\sum_i\int_0^t \sigma_i^2(s-t_i)1_{[t_i,t_{i+1})}(s)ds,$$
$$\mu(t,G)=\sum_i\int_0^t \nu_i(s-t_i)1_{[t_i,t_{i+1})}(s)ds+\sum_{s\in\Theta\cap[0,t]}\varepsilon_{-X(s-)}(G\setminus\{0\}),$$
$$\Gamma(t)=\sum_i\int_0^t \gamma_i(s-t_i)1_{[t_i,t_{i+1})}(s)ds-\sum_{s\in\Theta\cap[0,t]}X(s-)), t\geq 0\}$$

and the local characteristics are:

$$\{(\Theta, K(t,G), \sigma^2(t)=\sum_i \sigma_i^2(t-t_i)1_{[t_i,t_{i+1})}(t),$$
$$\nu(t)=\sum_i \nu_i(t-t_i)1_{[t_i,t_{i+1})}(t),$$
$$\gamma(t)=\sum_i \gamma_i(t-t_i)1_{[t_i,t_{i+1})}(t)), t\geq 0\}.$$

This is because we have

$$\{(\Theta=\{t_i\},K(t,G),A(t)=\int_0^t \sigma^2(s)ds=$$

$$=\sum_i \int_0^t \sigma_i^2(s-t_i)1_{[t_i,t_{i+1})}(s)ds,$$

$$\mu(t,G)=\int_0^t \nu(s,G)ds+\sum_{s\in\Theta\cap[0,t]}\mu(s,G)=$$

$$=\sum_i \int_0^t \nu_i(s-t_i)1_{[t_i,t_{i+1})}(s)ds+\sum_{s\in\Theta\cap[0,t]}K(s,G\setminus\{0\})=$$

$$=\sum_i \int_0^t \nu_i(s-t_i)1_{[t_i,t_{i+1})}(s)ds+\sum_{s\in\Theta\cap[0,t]}\varepsilon_{-X(s-)}(G\setminus\{0\}),$$

$$\Gamma(t)=\int_0^t \gamma(s)ds+\sum_{s\in\Theta\cap[0,t]}\Delta\Gamma(s)=$$

$$=\sum_i \int_0^t \gamma_i(s-t_i)1_{[t_i,t_{i+1})}(s)ds+\sum_{s\in\Theta\cap[0,t]}\int xK(s,dx)=$$

$$=\sum_i \int_0^t \gamma_i(s-t_i)1_{[t_i,t_{i+1})}(s)ds-\sum_{s\in\Theta\cap[0,t]}X(s-)),t\ge 0\}$$

In the case of Levy process with return to zero of type “termination” the point characteristics are

$$\{(\Theta=\{t_i\},K(t),A(t)=\int_0^t \sigma^2(s)ds=\sum_i \int_0^t \sigma_i^2 1_{[t_i,t_{i+1})}(s)ds=$$

$$=\sum_{t_i<t}(\min(t,t_{i+1})-t_i)\sigma_i^2,$$

$$\mu(t,G)=\sum_i \int_0^t \nu_i 1_{[t_i,t_{i+1})}(s)ds+\sum_{s\in\Theta\cap[0,t]}K(s,G\setminus\{0\})=$$

$$=\sum_{t_i<t}\nu_i(\min(t,t_{i+1})-t_i)+\sum_{s\in\Theta\cap[0,t]}K(s,G\setminus\{0\}),$$

$$\Gamma(t)=\sum_i \int_0^t \gamma_i 1_{[t_i,t_{i+1})}(s)ds+\sum_{s\in\Theta\cap[0,t]}\int xK(s,dx)=$$

$$=\sum_{t_i<t}\gamma_i(\min(t,t_{i+1})-t_i)+\sum_{s\in\Theta\cap[0,t]}\int xK(s,dx)),t\ge 0\}$$

and the local characteristics are

$$\{(\Theta=\{t_i\},K(t),\sigma^2(t)=\sum_i \sigma_i^2 1_{[t_i,t_{i+1})}(t),$$

$$\nu(t)=\sum_i \nu_i 1_{[t_i,t_{i+1})}(t),$$

$$\gamma(t)=\sum_i \gamma_i 1_{[t_i,t_{i+1})}(t),t\ge 0\}.$$

For Brownian motion without drift with returns to zero of type “termination” we have the integral characteristics

$$\{(\Theta=\{t_i\}, K(t), A(t)=\sum_{t_i<t}(\min(t,t_{i+1})-t_i)\sigma_i^2,$$
$$\mu(t,G)=\sum_{s\in\Theta\cap[0,t]}K(s,G\setminus\{0\}),$$
$$\Gamma(t)=\sum_{s\in\Theta\cap[0,t]}\int xK(s,dx)),t\geq 0\},$$

and the corresponding local characteristics are

$$\{(\Theta=\{t_i\}, K(t,G), \sigma^2(t)=\sum_i \sigma_i^2 1_{[t_i,t_{i+1})}(t),$$
$$\nu(t)=0, \gamma(t)=0), t\geq 0\}.$$

For Poisson process with return to zero of type “termination” in deterministic moments the integral characteristics are

$$\{(\Theta=\{t_i\}, K(t), A(t)=0,$$
$$\mu(t,G)=\sum_i \lambda_i(\min(t,t_{i+1})-t_i)f_i+\sum_{s\in\Theta\cap[0,t]}K(s,G\setminus\{0\}),$$
$$\Gamma(t)=\sum_{s\in\Theta\cap[0,t]}\int xK(s,dx)),t\geq 0\},$$

and the corresponding local characteristics are

$$\{(\Theta=\{t_i\}, K(t), \sigma^2(t)=0, \nu(t)=$$
$$=\sum_i \lambda_i f_i 1_{[t_i,t_{i+1})}(t), \gamma(t)=0), t\geq 0\}.$$

Additive processes with stochastic characteristics and returns to zero of type “termination” in deterministic moments

They are processes $X(t)=\sum_i G_i(t-t_i)1_{[t_i,t_{i+1})}(t),$ where $G_i(t)$ are additive processes with stochastic characteristics $A_i(\omega,t)$, $\mu_i(\omega,t,B)$ and $\Gamma_i(\omega,t)$ correspondingly, and t_i are the moments of returns to zero.

Additive processes with returns to zero of type “termination” in deterministic moments and stochastic characteristics have point characteristics:

$$\{(\Theta=\{t_i\}, K(\omega,t), A(\omega,t)=\sum_i A_i(\omega,t-t_i)1_{[t_i,t_{i+1})}(t),$$
$$\mu(\omega,t,B)=\sum_i \mu_i(\omega,t-t_i,B)1_{[t_i,t_{i+1})}(t)+\sum_{s\in\Theta\cap[0,t]}K(\omega,s,B\setminus\{0\}),$$
$$\Gamma(\omega,t)=\sum_i \Gamma_i(\omega,t-t_i)1_{[t_i,t_{i+1})}(t)+\sum_{s\in\Theta\cap[0,t]}\int xK(\omega,s,dx)),t\geq 0\}.$$

The set Θ is a discrete set giving the moments of the jumps. $K(t)$ is the conditional distribution of the jumps.

Again the retrn to zero is related to additional characterization of $K(t)$.

When the additive processes with retrns to zero of type “termination” are with local stochastic characteristics, the point characteristics are

$$\{(\Theta, K(\omega,t), A(\omega,t) = \sum_i \int_0^t \sigma_i^2(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds,$$

$$\mu(\omega,t,G) = \sum_i \int_0^t \nu_i(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds + \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(G \setminus \{0\}),$$

$$\Gamma(\omega,t) = \sum_i \int_0^t \gamma_i(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds - \sum_{s\in\Theta\cap[0,t]} \int X(s-)), t \ge 0\}$$

and the local characteristics are

$$\{(\Theta, K(\omega,t), \sigma^2(\omega,t) = \sum_i \sigma_i^2(\omega, t-t_i) 1_{[t_i,t_{i+1})}(t),$$

$$\nu(\omega,t) = \sum_i \nu_i(\omega, t-t_i) 1_{[t_i,t_{i+1})}(t),$$

$$\gamma(\omega,t) = \sum_i \gamma_i(\omega, t-t_i) 1_{[t_i,t_{i+1})}(t)), t \ge 0\}.$$

Proof: We have

$$\{(\Theta, K(\omega,t), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds = \sum_i \int_0^t \sigma_i^2(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds,$$

$$\mu(\omega,t,G) = \int_0^t \nu(\omega,s,G)ds + \sum_{s\in\Theta\cap[0,t]} \mu(\omega,s,G) =$$

$$= \sum_i \int_0^t \nu_i(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds + \sum_{s\in\Theta\cap[0,t]} K(\omega,s,G \setminus \{0\}) =$$

$$= \sum_i \int_0^t \nu_i(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds + \sum_{s\in\Theta\cap[0,t]} \varepsilon_{-X(s-)}(G \setminus \{0\}),$$

$$\Gamma(\omega,t) = \int_0^t \gamma(\omega,s)ds + \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(\omega,s) =$$

$$= \sum_i \int_0^t \gamma_i(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds + \sum_{s\in\Theta\cap[0,t]} \int xK(\omega,s,dx) =$$

$$= \sum_i \int_0^t \gamma_i(\omega, s-t_i) 1_{[t_i,t_{i+1})}(s)ds - \sum_{s\in\Theta\cap[0,t]} \int X(s-)), t \ge 0\}$$

In the case of Levy processes with returns to zero of type “termination” and stochastic characteristics the point characteristics are

$$\{(\Theta, K(\omega,t), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds =$$

$$= \sum_i \int_0^t \sigma_i^2(\omega) 1_{[t_i,t_{i+1})}(s)ds =$$

$$= \sum_{t_i<t} (\min(t,t_{i+1}) - t_i)\sigma_i^2(\omega),$$

$$\mu(\omega,t,G) = \sum_i \int_0^t \nu_i(\omega) 1_{[t_i,t_{i+1})}(s)ds + \sum_{s\in\Theta\cap[0,t]} K(\omega,s,G\setminus\{0\}) =$$

$$= \sum_{t_i<t} \nu_i(\omega)(\min(t,t_{i+1}) - t_i) + \sum_{s\in\Theta\cap[0,t]} K(\omega,s,G\setminus\{0\}),$$

$$\Gamma(\omega,t) = \sum_i \int_0^t \gamma_i(\omega) 1_{[t_i,t_{i+1})}(s)ds + \sum_{s\in\Theta\cap[0,t]} \int xK(\omega,s,dx) =$$

$$= \sum_i \gamma_i(\omega)(\min(t,t_{i+1}) - t_i) + \sum_{s\in\Theta\cap[0,t]} \int xK(\omega,s,dx)), t \ge 0\},$$

and the local characteristics are

$$\{(\Theta, K(\omega,t), \sigma^2(\omega,t) = \sum_i \sigma_i^2(\omega) 1_{[t_i,t_{i+1})}(t),$$

$$\nu(\omega,t) = \sum_i \nu_i(\omega) 1_{[t_i,t_{i+1})}(t),$$

$$\gamma(\omega,t) = \sum_i \gamma_i(\omega) 1_{[t_i,t_{i+1})}(t)), t \ge 0\}.$$

For Brownian motion without drift and returns to zero of type “termination” and stochastic characteristics the point characteristics are

$$\{(\Theta, K(\omega,t), A(\omega,t) = \sum_{t_i<t} (\min(t,t_{i+1}) - t_i)\sigma_i^2(\omega),$$

$$\mu(\omega,t,G) = \sum_{s\in\Theta\cap[0,t]} K(\omega,s,G\setminus\{0\}),$$

$$\Gamma(\omega,t) = \sum_{s\in\Theta\cap[0,t]} \int xK(\omega,s,dx)), t \ge 0\},$$

and the corresponding local characteristics are

$$\{(\Theta, K(\omega,t), \sigma^2(\omega,t) = \sum_i \sigma_i^2(\omega) 1_{[t_i,t_{i+1})}(t),$$

$$\nu(\omega,t) = 0, \gamma(\omega,t) = 0), t \ge 0\}.$$

For the Poisson process with returns to zero of type “termination” in deterministic moments and stochastic characteristics the integral characteristics are

$$\{(\Theta, K(\omega,t), A(\omega,t) = 0,$$

$$\mu(\omega,t,G) = \sum_i \lambda_i(\omega)(\min(t,t_{i+1}) - t_i) f_i + \sum_{s\in\Theta\cap[0,t]} K(\omega,s,G\setminus\{0\}),$$

$$\Gamma(\omega,t) = \sum_{s\in\Theta\cap[0,t]} \int xK(\omega,s,dx)), t \ge 0\},$$

an d the corresponding local characteristics are

$$\{(\Theta, K(\omega,t), \sigma^2(\omega,t) = 0, \nu(\omega,t) =$$

$$= \sum_i \lambda_i(\omega) f_i 1_{[t_i,t_{i+1})}(t), \gamma(\omega,t) = 0), t \ge 0\}.$$

Additive processes with stochastic characteristics and returns to zero of type "termination" at specific random times

The additive processes with stochastic characteristics and returns to zero of type "termination" at specific random times T_i we have the following point characteristics:

$$\{(\Theta(\omega,t)=\{T_i\},K(\omega,t),A(\omega,t)=$$
$$=\sum_i A_i(\omega,t-T_i)1_{[T_i,T_{i+1})}(t),$$
$$\mu(\omega,t,B)=\sum_i \mu_i(\omega,t-T_i,B)1_{[T_i,T_{i+1})}(t)+\sum_{T_i<t} K(\omega,T_i,B\setminus\{0\}),$$
$$\Gamma(\omega,t)=\sum_i \Gamma_i(\omega,t-T_i)1_{[T_i,T_{i+1})}(t)+\sum_{T_i<t_i}\int xK(\omega,T_i,dx)),t\geq 0\}.$$

Here $\{\Theta(\omega,t),t\geq 0\}$ is a point random process, giving the moments of jumps T_i, and $K(\omega,t)$ is the conditional distribution of the jumps for which we have

$$K(\omega,t,G)=\begin{cases}\mu(\omega,t,G)+\varepsilon_0(G)(1-\mu(\omega,t,R)),t\in\Theta\\ 0,t\notin\Theta.\end{cases}$$

In this case, some specifc jumps in times T_i are selected among the jumps and defined by the process $\{\Theta(\omega,t),t\geq 0\}$.

The returns to zero is related to additional characterization of $K(\omega,t)$.

When the additive processes with returns to zero of type "termination" and specific jumps at random times are with local stochastic characteristics, the integral characteristics

$$\{(\Theta(\omega,t),K(\omega,t),A(\omega,t)=$$
$$=\sum_{T_i<t}\int_{0_i}^{t}\sigma_i^2(\omega,s-T_i)1_{[T_i,T_{i+1})}(s)ds,$$
$$\mu(\omega,t,G)=\sum_{T_i<t}\int_{0_i}^{t}\nu_i(\omega,s-T_i)1_{[T_i,T_{i+1})}(s)ds+$$
$$+\sum_{T_i<t}K(\omega,T_i,G\setminus\{0\}),$$
$$\Gamma(\omega,t)=\sum_{T_i<t}\int_{0}^{t}\gamma_i(\omega,s-T_i)1_{[T_i,T_{i+1})}(s)ds+$$
$$+\sum_{T_i<t}\int xK(\omega,T_i,dx)),t\geq 0\}$$

and the local characteristics are

$$\{(\Theta(\omega,t),K(\omega,t),\sigma^2(\omega,t)=\sum_{T_i<t}\sigma_i^2(\omega,t-T_i)1_{[T_i,T_{i+1})}(t),$$
$$\nu(\omega,t)=\sum_{T_i<t}\nu_i(\omega,t-T_i)1_{[T_i,T_{i+1})}(t),$$
$$\gamma(\omega,t)=\sum_{T_i<t}\gamma_i(\omega,t-T_i)1_{[T_i,T_{i+1})}(t)),t\geq 0\}.$$

Indeed, we have

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds =$$

$$= \sum_{T_i<t} \int_{0_i}^t \sigma_i^2(\omega, s-T_i) 1_{[T_i,T_{i+1})}(s)ds,$$

$$\mu(\omega,t,G) = \int_0^t \nu(\omega,s,G)ds + \sum_{T_i<t} K(\omega,T_i,G\setminus\{0\}) =$$

$$= \sum_{T_i<t} \int_{0_i}^t \nu_i(\omega, s-T_i) 1_{[T_i,T_{i+1})}(s)ds + \sum_{T_i<t} K(\omega,T_i,G\setminus\{0\}),$$

$$\Gamma(\omega,t) = \int_0^t \gamma(\omega,s)ds + \sum_{T_i<t} \Delta\Gamma(\omega,T_i) =$$

$$= \sum_{T_i<t} \int_0^t \gamma_i(\omega, s-T_i) 1_{[T_i,T_{i+1})}(s)ds +$$

$$+ \sum_{T_i<t} \int xK(\omega,T_i,dx)), t \ge 0\}$$

In the case of Levy processes with stochastic characteristics and returns to zero of type “termination” at specific random moments the point characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) = \sum_{T_i<t} (\min(t,T_{i+1}) - T_i)\sigma_i^2(\omega),$$

$$\mu(\omega,t,G) = \sum_{T_i<t} \nu_i(\omega)(\min(t,T_{i+1}) - T_i) + \sum_{T_i<t} K(\omega,T_i,G\setminus\{0\}),$$

$$\Gamma(\omega,t) = \sum_{T_i<t} \gamma_i(\omega)(\min(t,T_{i+1}) - T_i) + \sum_{T_i<t} \int xK(\omega,T_i,dx)), t \ge 0\}$$

and the local characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), \sigma^2(\omega,t) = \sum_{T_i<t} \sigma_i^2(\omega) 1_{[T_i,T_{i+1})}(t),$$

$$\nu(\omega,t) = \sum_{T_i<t} \nu_i(\omega) 1_{[T_i,T_{i+1})}(t),$$

$$\gamma(\omega,t) = \sum_{T_i<t} \gamma_i(\omega) 1_{[T_i,T_{i+1})}(t)), t \ge 0\}.$$

For Brownian motion without drift and returns to zero of type “termination” at specific random moments and with stochastic characteristics the point characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) =$$

$$= \sum_{T_i<t} (\min(t,T_{i+1}) - T_i)\sigma_i^2(\omega),$$

$$\mu(\omega,t,G) = \sum_{T_i<t} K(\omega,T_i,G\setminus\{0\}),$$

$$\Gamma(\omega,t) = \sum_{T_i<t} \int xK(\omega,T_i,dx)), t \ge 0\},$$

and the local characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), \sigma^2(\omega,t) = \sum_{T_i<t} \sigma_i^2(\omega) 1_{[T_i,T_{i+1})}(t),$$

$$\nu(\omega,t) = 0, \gamma(\omega,t) = 0), t \ge 0\}.$$

For the Poisson process with retrns to zero of type “termination” in specific random moments and stochastic characteristics the point characteristics are

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t) = 0,$$
$$\mu(\omega,t,G) = \sum_{T_i<t} \lambda_i(\omega)(\min(t,T_{i+1}) - T_i) f_i +$$
$$+ \sum_{T_i<t} K(\omega,T_i,G \setminus \{0\}),$$
$$\Gamma(\omega,t) = \sum_{T_i<t} \int xK(\omega,T_i,dx)), t \geq 0\},$$

and the corresponding local characteristics are:

$$\{(\Theta(\omega,t), K(\omega,t), \sigma^2(\omega,t) = 0, \nu(\omega,t) =$$
$$= \sum_{T_i<t} \lambda_i(\omega) f_i 1_{[T_i,T_{i+1})}(t), \gamma(\omega,t) = 0), t \geq 0\}.$$

Switch Time distributions (Stoynov distributions) - STF(n,β)

We say that a random variable ξ with probability mass function $f_\xi(x)$ has distribution of $ST(n,\beta)$ family and denote this fact $\xi \in STF(n,\beta)$, if the probability mass function of ξ is given by the formula

$$f_\xi(x) = \begin{cases} \sum_{k=1}^{n+1} P(D^n = k) f_\xi(x \mid D^n = k) = \sum_{k=1}^{n+1} P(D^n = k) f_{G^k}(x), x \geq 0 \\ 0, x < 0. \end{cases}$$

where G^k are random variables with probability mass function $f_{G^k}(x) = f(k,\beta)$ and D^n are random variables taking values $k = 1,\ldots,(n+1)$ with probabilities $P(D^n = k) = \frac{C(n,\beta)n!}{\beta^k (n-k+1)!}, k = 1,\ldots,(n+1)$ where the coefficients $C(n,\beta)$ are given by the formulas:

$$C(n,\beta) = \frac{1}{I(n,\beta)},$$
$$I(0,\beta) = \frac{1}{\beta},$$
$$I(n,\beta) = \frac{1}{\beta} + \frac{n}{\beta} I(n-1,\beta), n = 1,2,\ldots$$

Also, variables $\tilde{D}^n = D^n - 1$ can be introduced taking values $k = 0,\ldots,n$ with probabilities

$$P(\tilde{D}^n = k) = \frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!}, k = 0,\ldots,n.$$

Then the probability mass function $f_\xi(x)$ of ξ may be presented also by the formula

$$f_\xi(x) = \begin{cases} \sum_{k=0}^{n} P(\tilde{D}^n = k) f_\xi(x \mid \tilde{D}^n = k) = \sum_{k=0}^{n} P(\tilde{D}^n = k) f_{G^{k+1}}(x), x \geq 0 \\ 0, x < 0. \end{cases}$$

Here for G^k we may adopt different families of distribution. For example, if we take $G^k \in \Gamma(k,\frac{1}{\beta}) \equiv Erlang(k,\frac{1}{\beta})$, i. e. $\xi \mid D^n \equiv Erlang(D^n,\frac{1}{\beta})$, we obtain a distribution which we will call $ST1(n,\beta)$ distribution, having probability density function

$$f_{\xi}(x)=\begin{cases}C(n,\beta)e^{-\beta x}(1+x)^{n},\ x\geq 0\\ 0, x<0\end{cases}.$$

Indeed, we have:

$$C(n,\beta)e^{-\beta x}(1+x)^{n}=C(n,\beta)e^{-\beta x}\sum_{k=0}^{n}\binom{n}{k}x^{k}=\sum_{k=0}^{n}C(n,\beta)\binom{n}{k}x^{k}e^{-\beta x}=$$

$$=\sum_{k=0}^{n}\frac{C(n,\beta)n!x^{k}e^{-\beta x}}{k!(n-k)!}=\sum_{k=0}^{n}\frac{C(n,\beta)n!\beta^{k+1}x^{k}e^{-\beta x}}{\beta^{k+1}(n-k)!k!}=$$

$$=\sum_{k=0}^{n}\frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!}\frac{\beta^{k+1}x^{k}e^{-\beta x}}{k!}=$$

$$=\sum_{k=0}^{n}P(D^{n}=k+1)f_{G^{k+1}}(x)=$$

$$=\sum_{k=1}^{n+1}\frac{C(n,\beta)n!\beta^{k}x^{k-1}e^{-\beta x}}{\beta^{k}(n-k+1)!(k-1)!}=$$

$$=\sum_{k=1}^{n+1}\frac{C(n,\beta)n!}{\beta^{k}(n-k+1)!}\frac{\beta^{k}x^{k-1}e^{-\beta x}}{\Gamma(k)}=\sum_{k=1}^{n+1}\frac{C(n,\beta)n!}{\beta^{k}(n-k+1)!}\frac{\beta^{k}x^{k-1}e^{-\beta x}}{(k-1)!}=$$

$$\sum_{k=1}^{n+1}P(D^{n}=k)f_{G^{k}}(x).$$

Special cases of $ST1(n,\beta)$ distribution are $ST1(0,\beta)\equiv Exp(\beta)$ and $ST1(1,\beta)\equiv Lindley(\beta)$. The case $n=2$ is introduced in Stoynov (2010) and further generalized in some other publications of the author, for example Stoynov (2011) where Moment Generating Function of $ST1(2,\beta)$ is derived.

As another example we may consider negative binomial distribution for G^{k}, i. e. $G^{k}\in NB(k,e^{-\beta})$. In this case, $\xi\,|\,D^{n}\equiv NB(D^{n},e^{-\beta})$ and we obtain a distribution which we will call $ST2(n,\beta)$ distribution.

Special cases of $ST2(n,\beta)$ distribution are $ST2(0,\beta)\equiv NB(1,e^{-\beta})$ and $ST2(1,\beta)\equiv\frac{\beta}{\beta+1}NB(1,e^{-\beta})+\frac{1}{\beta+1}NB(2,e^{-\beta})$.

We say that a random variable ξ has a negative binomial distribution (Pólya distribution) and write this $\xi\in NB(r,p)$, if its probability satisfies

$$P(\xi=k)=\binom{r+k-1}{k}p^{r}(1-p)^{k},\ k=0,1,2,\ldots$$

We may interpret this random variable as a counter of the failures till the $r-th$ success in a sequence of Bernoulli trials.

As a third example we may consider $G^{k}\equiv\delta_{k}(x)$ - random variable which takes value k with probability one. Then $\xi\equiv D^{n}\in D^{n}(\beta)$. In this case, we say that random variable ξ has $ST3(n,\beta)$ distribution and denote $\xi\in ST3(n,\beta)$.

Special cases of $ST3(n,\beta)$ distribution are $ST3(0,\beta)\equiv D^{0}\equiv 1$ and $ST3(1,\beta)\equiv D^{1}$.

ST(n,β) processes

We say that a process $X(t)$ is $ST1(n,\beta)$ ($ST2(n,\beta)$) and denote this fact $X(t)\in ST1(t;n,\beta)$ ($X(t)\in ST1(t;n,\beta)$), if it has the following properties:

1) $X(0)=0$.

2) The process has independent increments, i. e. for

$$0 < t_1 < t_2 < \cdots < t_n$$

the random variables

$$X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent.

3) The process is continuous stochastically (continuous in probability), i. e.

$$\lim_{t \to s} X(t) = X(s),$$

where the limit is taken in probability.

4) $X(t)$ is pure jump process with jumps at times $T_{i'}, i = 1,2,\ldots$ and jump sizes $\Delta X(T_i) = 1.$

5) The intervals between two jumps are

$$\tau = T_i - T_{i-1} \in ST1(n-1, \beta)\,(ST2(n-1, \beta)), i = 0,1,\ldots, T_0 = 0$$

The first three conditions mean that $ST1(n, \beta)$ ($ST2(n-1, \beta)$) process is an additive process. If we replace condition 4) with condition

4') $X(t)$ is pure jump process with jumps at times $T_{i'}, i = 1,2,\ldots$ and jump sizes $\Delta X(T_i) = Y_i$, where Y_i are independent and identically distributed random variables,

we say that $X(t)$ is compound $ST1(n, \beta)$ ($ST2(n, \beta)$) process.

It is interesting to see for $ST(n, \beta)$ processes the distribution of the number of jumps till time t which we denote $N^n(t)$.

We start with $ST1(n, \beta)$ process. When $X(t) \in ST1(t; n, \beta)$, we consider first the case $X(t) \in ST1(t; 1, \beta)$. In this case, $\tau = T_i - T_{i-1} \in ST1(0, \beta) \equiv Exp(\beta), i = 0,1,\ldots, T_0 = 0$. In this case the process $N^1(t)$ is a Poisson process $N(t)$ and for it we have

$$P(N^1(t) = k) = \frac{(\beta t)^k}{k!} e^{-\beta t}.$$

In the case when $X(t) \in ST1(t; n, \beta)$ the intervals between two jumps are

$$\tau = T_i - T_{i-1} \in ST1(n-1, \beta) \equiv Erlang(D^n, \beta), i = 0,1,\ldots, T_0 = 0.$$

First we will consider the case where

$$\tau = T_i - T_{i-1} \in Erlang(n, \beta), i = 0,1,\ldots, T_0 = 0.$$

In this case, we have that the process which counts the jumps of an Erlang process till time t which we denote $N^E(t)$ is a Poisson process $N(t)$ with omitted $n-1$ jumps and actual jumps at every $n-$th jump of the Poison process, so

$$P(N^E(t) = k) = P(N(t) = kn) = \frac{(\beta t)^{kn}}{(kn)!} e^{-\beta t}.$$

So, in the case $X(t) \in ST1(t; n, \beta)$ we have

$$P(N^n(t) = k) = P(N(t) = kn) = \frac{(\beta t)^{kn}}{(kn)!} e^{-\beta t}.$$

Simulation of ST(n,β) distribution

The simulation of $ST(n, \beta)$ distribution and corresponding PMF graphics for different values of the parameters n and β can be done by using R language. Here is the code for creating graphic representation of the PMF (for $n = 100$ and $\beta = 2$):

```
#ST(n,beta) graphics
beta<-2
n<-100
```

```
In_beta<-function(n,beta, recursive=TRUE)
{if (n==0) return (1/beta) else return (1/beta+n/beta*In_beta(n-1,beta))}
Cn_beta<-function(n,beta) {1/In_beta(n,beta)}
STn_beta<-function(x,n,beta){Cn_beta(n,beta)*(1+x)^n*exp(-x*beta)}
m<-10000
STn_beta_values<-0:m
T<-100
tvec<-T*0:m/m
for (i in 1:m) {
        STn_beta_values[i]<-STn_beta(i*T/m,n,beta)
}
freq<-10   # take out every 1000th point in the simulation & plot it
pick<-freq*(0:(m/freq))+1
title<-paste("ST(", round(n,1), ",",round(beta,1), ")")
plot(tvec[pick],STn_beta_values[pick],type='l',xlab="x",ylab="f(x)",main=title,ylim=c(0,0.2))
```

The graphics for the case $n = 100$ and $\beta = 2$ is given in Fig 1. The graphics for the case $n = 40$ and $\beta = 2$ is given in Fig 2.

5. Simulation of ST(n,β) processes.

To make simulation of $ST(n,\beta)$ process, the following algorithm can be applied:

1. Define interval $[0,T]$ of the simulation.
2. Set $k = 0.$
3. While $\sum_{i=1}^{k} \tau_i < T$ do:
   - 3.1. Set $k = k + 1$
   - 3.2. Generate $\tau_k \in ST(n-1,\beta).$
   - 3.3. Set $Y_k = 1$ for standard $ST(n,\beta)$ process or simulate Y based on a given distribution f for compound $ST(n,\beta)$ process.

Then the trajectory of $X(t) \in ST(t;n,\beta)$ is given by the formula

$$X(t) = \sum_{i=1}^{N(t)} Y_i,$$

where

$$N(t) = \sum_i 1_{\{T_i < T\}}.$$

Here is the code for creating graphic representation of the $ST(n,\beta)$ process:

```
#ST(n+1,beta)_process simulation
# Theta_grid_sets Poisson, Erlang, ST
T<-1000
lambda<-0.1
Theta_Poisson <-rexp(1,lambda)
while(max(Theta_Poisson)<T) Theta_Poisson <-c(Theta_Poisson,max(Theta_Poisson)+rexp(1,lambda))
m<-length(Theta_Poisson)
Theta_Poisson[m]<-T

# Erlang
beta_ER<-lambda
n_ER<-2
m_ER<-m/n_ER
Theta_ER<- Theta_Poisson[n_ER]
for (i in 2:m_ER)
{
Theta_ER<- c(Theta_ER, Theta_Poisson[i*n_ER])
}
```

```
#ST
ST_n<-5
ST_beta<-0.1
n<-20
tvec<-T*0:n/n
In_beta<-function(n,beta, recursive=TRUE)
{if (n==0) return (1/beta) else return ((1/beta)+(n/beta)*In_beta(n-1,beta))}
Cn_beta<-function(n,beta) {1/In_beta(n,beta)}
FDn_inverse<-function(y,n,beta){
Dn_values<-1:(n+1)
Dn_prob<-1:(n+1)
for (i in 1:(n+1))
{
Dn_prob[i]<-(Cn_beta(n,beta)*factorial(n)/(beta^i*factorial(n-i+1)))
}
Prob_acum<-Dn_prob[1]+Dn_prob[2]
for (k in 1:n)
{
if (y< Prob_acum) return (k)
else
Prob_acum<-Prob_acum+Dn_prob[k+2]
}
return(n+1)
}


#Dn_simulation<-function(n,ST_n,ST_beta){
Dn_values<-1:(n+1)
Dn_prob<-1:(n+1)
Dn_simulated<-1:n
for (i in 1:(n+1))
{
Dn_prob[i]<-(Cn_beta(n,ST_beta)*factorial(n)/(ST_beta^i*factorial(n-i+1)))
}
for (i in 1:n)
{
St_unif_value<- runif(1, min = 0, max = 1)
Dn_simulated[i]<-FDn_inverse(St_unif_value,ST_n,ST_beta)
}
acc_ST<- Dn_simulated[1]

Theta_ST<- Theta_Poisson[acc_ST]

while(acc_ST<m)
{
acc_ST<-(acc_ST+Dn_simulated[i])
if (acc_ST<=m)
Theta_ST<- c(Theta_ST, Theta_Poisson[acc_ST])
}
Theta_ST
s<-8000
tvec<-T*0:s/s
N<-0:s
N[1]<-0
u<-length(Theta_ST)

for (i in 1:s)
{
```

```
            N[i]<-0
            for (j in 1:u)
            {
                        if ((T*(i-1)/s)> Theta_ST[j]) N[i]<-(N[i]+1)
            }
}
title<-paste("Траектория на ST процес n=",round((ST_n+1),1), ", beta=", round(ST_beta,2))
plot(tvec[1:(s-1)],N[1:(s-1)],type='l', xlab="t",ylab="N(t)", main=title)
```

The graphics of a trajectory of $ST(n,\beta)$ process for the case $n=6$ and $\beta=0.1$ is given in Fig 3.

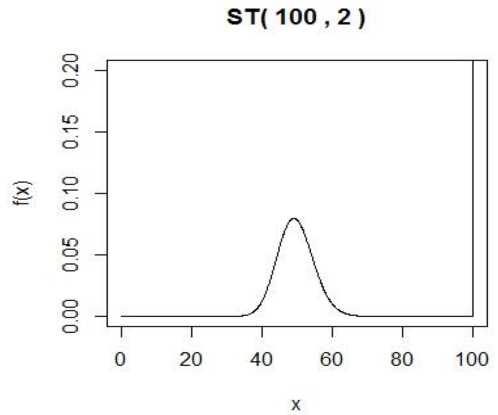

Figure 1. The graphics of $ST(n,\beta)$ distribution for the case $n=100$ and $\beta=2$.

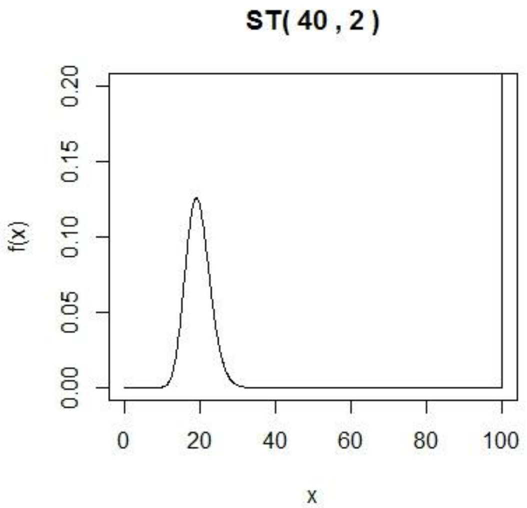

Figure 2. The graphics of $ST(n,\beta)$ distribution for the case $n=40$ and $\beta=2$.

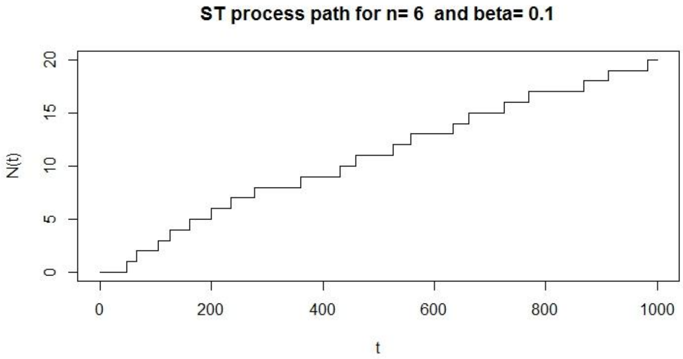

Figure 3. The graphics of a path of $ST(n,\beta)$ process for the case $n=6$ and $\beta=0.1$.

Conclusion

Switch time distributions presented in the article posses some suitable properties for modeling more complicated switch time structures in additive processes and their generalizations.

The present work may be extended for typical (not degenerated) ST processes as well as by studying other choices of G^k.

For example, we may choose $G^k \in NB(k,e^{-\beta})$. In this case, $\xi \mid D^n \equiv NB(D^n,e^{-\beta})$ and we obtain a distribution which we will call $ST2(n,\beta)$ distribution.

As another example we may consider $G^k \equiv \delta_k(x)$ - random variable which takes value k with probability one. Then $\xi \equiv D^n \in D^n(\beta)$. In this case, we say that random variable ξ has $ST3(n,\beta)$ distribution and denote $\xi \in ST3(n,\beta)$.

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