A note on the time evolution of bivariate copulas

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Abstract

Dependence properties between random events are important issues not only for researches but also from the viewpoint of risk management. For example, the dependence relation between exchange rates is worth further investigation both for academics and for insurance industries. To discuss such association problems, Kendall's tau τ and Spearman's rho ρ are known measures of concordance between random variables. On the other hand, a copula is well employed tool for analyzing dependence relations, and the formula of τ and ρ in terms of copula is already known. In our previous study, we introduce the time evolution of bivariate copulas, which is provided as a solution of diffusion equation. This paper is concerned with the convergence in time of τ and ρ under the evolution of copulas. It is shown that any dependence structure converges to the one of independence as the time tends to infinity.

Keywords: copula, bivariate copula, risk management

1. Introduction

Dependence relations between random variables are one of the most important subjects for researches in probability and statistics. For example, the dependence between exchange rates, such as the rate between Japanese yen (JY) against US dollar and JY against Euro is worth intensive

2000 *Mathematics Subject Classification.* 91G80; 35K05.

Key words: copula, time-parametrized family, Kendall's tau, Spearman's rho, convergence

investigation. To mention another, the damage caused by typhoon is randomly related between Taiwan and Japan; some typhoon move from Taiwan to Japan but others do not. The insurance company whose business includes these two countries wants to estimate the effect of this dependence. To quantitatively estimate such dependences, several measures of associations have introduced so far. As widely known examples, we recall the population version of Kendall's tau and Spearman's rho, whose explicit formulation will be provided shortly later.

On the other hand, a copula is introduced as a tool for understanding the dependence structure among random variables. Copulas make a link between multivariate joint distributions and univariate marginal distributions. The definition of copula and the fundamental theorem due to A. Sklar [6] is expressed as follows in the case of a bivariate joint distribution.

Definition 1. A function C defined on $I^2 := [0,1] \times [0,1]$ and valued in I is called a copula if the following conditions are fulfilled.

(i) For every $(u,v) \in I^2$

$$C(u,0) = C(0,v) = 0, \quad C(u,1) = 1 \text{ and } C(1,v) = v.$$

(ii) For every $(u_i, v_i) \in I^2 (i = 1,2,)$ with $u_1 \leq u_2$ and $v_1 \leq v_2$

(2) $$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0$$

The requirement (2) is referred to as *the* 2-*increasing condition*. We also note that a copula is continuous by its definition

Theorem 2. (Sklar's theorem) *Let H be a bivariate joint distribution function with marginal distribution functions F and G; that is,*

$$\lim_{x \to \infty} H(x,y) = G(y), \qquad \lim_{y \to \infty} H(x,y) = F(y).$$

Then there exists a copula, which is uniquely determined on $RanF \times RanG$ *, such that*

(3) $$H(x,y) = C(F(x), G(y)).$$

Conversely, if C *is a copula and F and G are distribution functions, then the function* H *defined by (3) is a bivariate joint distribution function with marginals F and G..*

The population version of Kendall's tau and Spearman's rho, which are denoted by τ and ρ, respectively, are known to be represented in terms of copulas. See for instance an excellent monograph by R.B. Nelsen [4]. Precisely stated, let X and Y be continuous random variables whose copula is C. Then we have

$$\tau_{X,Y} = \tau_c = 4 \iint_{I^2} C(u,v)\, dC(u,v) - 1 = 1 - 4 \iint_{I^2} \frac{\partial C}{\partial u}(u,v) \frac{\partial C}{\partial v}(u,v)\, du\, dv$$

$$\rho_{X,Y} = \rho_c = 12 \iint_{I^2} C(u,v) du\, dv - 3 = 12 \iint_{I^2} (C(u,v) - uv) du\, dv$$

Now, in our previous study [3], we have introduced the time evolution of copulas. That is, we consider a time parameterized family of copulas $\{C(u,v,t)\}_{t \geq 0}$, which satisfy the heat equation:

(4) $$\frac{\partial C}{\partial t}(u,v,t) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) C(u,v,t).$$

Here, by the definition of copula, we understand that $C(\cdot,\quad \cdot,t)$ fulfills (1)(2); to be precisely, we postulate that

(i) for every $(u,v,t)\in I^2\times(0,\infty)$,

(5) $$C(u,0,t)=C(0,v,t)=0,\qquad C(0,1,t)=u,\qquad C(1,v,t)=v.$$

(ii) for every $(u_i,v_i,t)\in I^2\times(0.\infty)(i=1,2,)$ with $u_1\le u_2$ and $v_1\le v_2$

(6) $$C(u_1,v_1,t)-C(u_1,v_2,t)-C(u_2,v_1,t)+C(u_2,v_2,t)\ge 0.$$

The stationary solution to (4), which is referred to as the harmonic copula, is uniquely determined to be $\Pi(u,v):=uv$ in view of the boundary condition (1). We note that the copula Π represents the independent structure between two respective random variables.

The primal establishment of [3] is the existence proof of solutions to (4), which satisfy the copula conditions (5)(6). In particular, the solution *u* = *u*(*u*, *v*, *t*) is expressed as

$C(u,v,t)=$

(7)

$$uv+4\sum_{m,n=1}^{\infty}e^{-\pi^2(m^2+n^2)t}\sin m\pi u\sin n\pi v\iint_{I^2}\sin m\pi\xi\,\sin n\pi\eta\,\left(C_0(\xi,\eta)-\xi\eta\right)d\xi\,d\eta$$

In this article, we are concerned with the convergence as $t\to\infty$ of solutions to (4). Since $C=C(u,v,t)$ verifies the diffusion equations, it is natural to expect that C convergex exponentially as $t\to\infty$ in some function spaces. Indeed, we explicitly show the exponential convergence of C as well as τ_C and ρ_C

Now the main theorem of this paper reads as follows.

Theorem 3. *Let be a time-parametrized family of bivariate copulas, which satisfy*

(8) $$\frac{\partial C}{\partial t}(u,v,t)=\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)C(u,v,t).\ \text{ for }\ (u,v,t)\in I^2\times(0,\infty),$$

$$C(u,v,0)=C_0(u,v)\qquad \textbf{on } (u,v)\in I^2$$

where C_0 *denotes a given initial copula. Then it follows that*

$$\left|\tau_{C_t}\right| + \left|\rho_{C_t}\right| \leq Ae^{-Bt} \quad \text{as} \quad t \to \infty$$

where A and B are positive constants.

In particular, $C_t \to \Pi$ exponentially as $t \to \infty$.

In the next section, we prove this theorem. Discussions are given in the final section

2. Proof of Main Theorem

We employ the notation: $\|(C-\Pi)(t)\|_2^2 := \iint_{I^2} (C(u,v,t) - uv)^2 du\, dv$. The proof proceeds with elementary estimates.

We first compute

$$\frac{1}{2}\frac{d}{dt}\|(C-\Pi)(t)\|_2^2 = \iint_{I^2} (C-\Pi)\Delta(C-\Pi) du\, dv = -\|\nabla(C-\Pi)(t)\|_2^2 \tag{9}$$

$$\leq -\alpha\|(C-\Pi)(t)\|_2^2$$

where $\alpha > 0$ denotes a constant due to the Poincaré inequality. The use of this inequality is possible thanks to. We thus obtain

$$\|(C-\Pi)(t)\|_2^2 \leq \|(C_0-\Pi)\|_2^2 e^{-2\alpha t}$$

Taking into account of this estimate, we infer that

$$\left|\rho_{C_t}\right| = \left|12\iint_{I^2} (C-\Pi) du\, dv\right| \leq 12\|(C_0-\Pi)\|_2 e^{-\alpha t}$$

This proves the exponential convergence of ρ .

Next we turn our attention to τ . We recall the next formula, which is derived upon direct calculation.

$$\tau_c = \frac{2}{3}\rho_c - 4\iint_{I^2} \frac{\frac{\partial}{\partial u(C(u,v) - uv)}\partial}{\partial v(C(u,v) - uv)du} dv$$

It then follows that

$$e^{\alpha t}|\tau_C(t)| \leq \frac{2}{3}e^{\alpha t}|\rho_C(t)|$$

$$\leq 8\|(C_0-\Pi)\|_2 + 4e^{\alpha t}\|\nabla(C-\Pi)(t)\|_2^2$$

In light of the continuity of $\|\nabla(C-\Pi)(t)\|_2$ with respect to *t*, it suffices to show that

$$\int_t^\infty e^{\alpha s}\|\nabla(C-\Pi)(s)\|_2^2 ds < \infty$$

which implies that $e^{\alpha t}\|\nabla(C-\Pi)(t)\|_2^2 \leq M < \infty$ for all large *t* with some constant *M*.

In view of (9), we infer that

$$\frac{\alpha}{2}e^{\alpha t}\|(C-\Pi)(t)\|_2^2$$
$$\leq \frac{\alpha}{2}e^{-\alpha t}\|(C_0-\Pi)\|_2^2 ,$$

from which we conclude that

$$2\int_t^\infty e^{\alpha s}\|\nabla(C-\Pi)(s)\|_2^2\, ds \leq e^{\alpha t}\|(C-\Pi)(t)\|_2^2+\int_t^\infty \alpha e^{-\alpha s}\|(C_0-\Pi)\|_2^2\, ds$$
$$\leq 2\|(C_0-\Pi)(s)\|_2^2 e^{-\alpha t}.$$

In summary, we have established $e^{\alpha t}|\tau_C(t)| \leq 8\|(C_0-\Pi)\|_2 + 4M$ and the proof of the exponential convergence of τ is thereby completed.

3. Discussions

We have established the convergence of time-dependent family of copulas $\{C_t\}_{t\geq 0}$, which satisfy the heat equation. It is shown that C_t converges exponentially to the product copula Π as the time tends to infinity; the corresponding Kendall's tau τ_{C_t} and the Spearman's rho ρ_{C_t} converge exponentially to zero as $t \to \infty$. Since our copulas verify the diffusion equation, dependence relation converges to the standard one, namely, the independence relation.

Copulas are well investigated in these days. Because of their flexible structure, copulas have been applied in many situations. We refer for instance to W.F. Darsow et al. [1] and E.W. Frees and E.A. Valdez [2]. For other materials, we refer to the well summarized review articles of H. Tsukahara [7] and Y.Yoshizawa [8][9].

Time dependent copulas also have been studied so far, which is introduced by A.J. Patton [5] and is referred to as dynamic copulas. See also [10]. Typical procedure of constructing dynamic copulas is as follows: There exists a class of one-parameter families of copulas, known as the Archimedean copulas (see [4]). We take one such copula and denote it by $C(u,v;\,\rho)$, where ρ is a parameter whose value should belong to some interval $J(\subset \mathbb{R})$. The dynamic copula is then provided as

$$C(u,v;\,\rho_t) \text{ with } \rho_t = \Lambda(X_t(\rho_{t-1}))$$

where $X_t(\rho_{t-1})$ means some time-series model, say ARMA(*p*, *q*)-type process, and Λ is the transformation function which is designed to keep $\rho_t \in J$.

Compared to the concept of dynamic copulas, in our time-dependent model, copula itself evolves according to the heat equation. Our evolving copulas eventually converge to the sta- tionary copula, namely, the product copula. Such phenomena are rather commonly observed in diffusion processes.

From practical point of view, the maturity condition is much appropriate than the initial value setting as we did here. As an example of this version, we recall the next theorem which is exhibited in [3]

Theorem 4. *For any bivariate copula* $C_T(u,v)$, *where* T (> 0) *denotes the maturity, there* *exists a unique family of time-parametrized bivariate copulas* $\{C(u,v,t)\}_{0\le t\le T}$ *such that*

$$\frac{\partial C}{\partial t}(u,v,t)+\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)C(u,v,t)=0$$
$$C(u,v,T)=C_T(u,v) \qquad \text{on } (u,v)\in I^2$$

Here the convergence property as $t\to -\infty$ is not so relevant issue from the financial standpoint.

Acknowledgement

The first author (NI) is supported in part by the grant from the Japan Society for the Promotion of Sciences (No.21540117).

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