Additive processes – characterization possible generalizations, specific types and simulation

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Abstract.
A parameterization of additive processes is presented. A family of probability distributions called switch time family is introduced - STF(n,β) - with two representatives ST1(n,β) distribution and ST2(n,β) distribution. Corresponding stochastic processes are defined and studied. Simulation and graphics of the distributions and the processes in R languages are done as well as some applications.

Keywords: additive processes; parameterization; switch time family distributions and processes; simulation

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1. Additive processes – characterization and possible generalisations.

With a given filtered probability space (Ω, F, F_t, P) satisfying the usual conditions, an (one or d-dimensional) additive process is defined as a stochastic process $\{X(t), t \geq 0\}$ which is càdlàg and satisfies the following conditions:

1. $X(0) = 0;$
2. The process has independent increments, i. e. for every sequence of numbers

$$0 < t_1 < t_2 < \cdots < t_n$$

the random variables

$$X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent;

3. The process is stochastically continuous (continuous in probability), i. e.

$$\lim_{t \to s} X(t) = X(s),$$

where the limit is taken by probability.

The additive processes are introduced by Lévy (1937, 1992). Their properties are studied in Sato (1999). In Sato (1999), the following theorem is presented:

Theorem (Sato) Let $\{X(t), t \geq 0\}$ be an additive process with values in R^d. Then:

1. For every $t \geq 0$ the random variable $X(t)$ has infinitely divisible distribution.
2. The distribution of $\{X(t), t \geq 0\}$ is uniquely identified by its point characteristics $\{(A(t), \mu(t), \Gamma(t)), t \geq 0\}$, which are related to the characteristic function in the following way:

$$Ee^{iuX(t)} = e^{\psi(t,u)}$$

and

$$\psi(t,u) = -\frac{1}{2}uA(t)u + iu\Gamma(t) + \int_{R^d} \mu(t,dx)(e^{iux} - 1 - iux1_{\{|x| \leq 1\}}).$$

The point characteristics of the additive process satisfy the properties:

2.1. For every $t \geq 0$ $A(t)$ is positive definite matrix with dimension $d \times d$, $\Gamma(t)$ is a function with values which are vectors in R^d and $\mu(t,dx)$ is a positive measure in R^d, satisfying the conditions $\mu(t,\{0\}) = 0$ and $\int_{R^d} \min(|x|^2, 1)\mu(t,dx) < \infty.$

2.2. $A(0)=0, \mu(0,dx)=0, \Gamma(0)=0$ and for all $t \geq s \geq 0$ the matrix $A(t)-A(s)$ is a positive definite matrix with dimension $d \times d$ and $\mu(s,B) \leq \mu(t,B)$ for all measurable sets $B \in \mathrm{B}(R^d)$.

2.3. If $s \to t$, then $A(s) \to A(t), \Gamma(s) \to \Gamma(t), \mu(s,B) \to \mu(t,B)$ for all measurable sets $B \in \mathrm{B}(R^d)$, for which $B \subset \{x : |x| \geq \varepsilon\}$ for some $\varepsilon > 0$.

Conversely, for the family triples $\{(A(t), \mu(t), \Gamma(t)), t \geq 0\}$, which satisfies 2.1., 2.2. and 2.3. there is an additive process $\{X(t), t \geq 0\}$ with the corresponding point characteristics.

Important special cases of the additive processes are the processes for which there exists the following parametric representation of the point characteristics:

1. $A(t) = \int_0^t \sigma^2(s)ds$, where for every $t \geq 0$ the matrix $\sigma(t)$ is a real matrix with dimension $d \times n$ for which $\int_0^T \sigma^2(t)dt < \infty$ for fixed $T > 0$ such that $\sigma(t)$ is defined in the interval $[0,T]$.

2. $\mu(t,B) = \int_0^t \nu(s,B)ds$ for all measurable sets $B \in \mathrm{B}(R^d)$, where $\{\nu(t), t \in [0,T]\}$ is a family of Lévy measures satisfying the condition $\int_0^T \int_{R^d} \min(|x|^2, 1)\nu(t,dx)dt < \infty$.

3. $\Gamma(t) = \int_0^t \gamma(s)ds$, where $\gamma : [0,T] \to R$ is a deterministic function with finite variation.

For the triple $(\sigma^2(t), \nu(t), \gamma(t))$ we say that it defines the local characteristics of the additive process. The additive processes with local characteristics are semimartingales.

There are different possibilities for generalization of the additive processes.

One way of generalization is to introduce jumps in deterministic moments, as proposed by Kallsen (1998). In this way, the process is not already stochastically continuous and with stationary increments. To characterize such kind of processes, to the point and local characteristics there is a need to add new characteristics identifying the position and the size of these deterministic-time jumps.

The point characteristics become

$$\{(\Theta, K(t), A(t), \mu(t), \Gamma(t)), t \geq 0\},$$

where Θ is a discrete set giving the times of the jumps and $K(t)$ is conditional distribution of the jumps for which

$$K(t,G) = \begin{cases} \mu(t,G) + \varepsilon_0(G)(1-\mu(t,R)), t \in \Theta \\ 0, t \notin \Theta. \end{cases}$$

Here $\varepsilon_0(G)$ is the Dirac measure in the point 0.

In the case when local characteristics exist we have

$$\{(\Theta, K(t), A(t) = \int_0^t \sigma^2(s)ds,$$

$$\mu(t,G) = \int_0^t \nu(s,G)ds + \sum_{s \in \Theta \cap [0,t]} \mu(s,G) = \int_0^t \nu(s,G)ds + \sum_{s \in \Theta \cap [0,t]} K(s, G \setminus \{0\}),$$

$$\Gamma(t) = \int_0^t \gamma(s)ds + \sum_{s \in \Theta \cap [0,t]} \Delta\Gamma(s) = \int_0^t \gamma(s)ds + \sum_{s \in \Theta \cap [0,t]} \int xK(s,dx)), t \geq 0\},$$

and the local characteristics are:

$$\{(\Theta, K(t), \sigma^2(t), \nu(t), \gamma(t)), t \geq 0\}.$$

A next step of generalization is to allow stochastic characteristics- drift, volatility and jump intensity. The corresponding processes are considered for example in Grigelionis (1973).

In this case, the point characteristics of the process are

$$\{(\Theta, K(\omega,t), A(\omega,t), \mu(\omega,t), \Gamma(\omega,t)), t \geq 0\}$$

and the local characteristics (when they exist) are

$$\{(\Theta, K(\omega,t), \sigma^2(\omega,t), \nu(\omega,t), \gamma(\omega,t)), t \geq 0\}.$$

The obtained characterization allows to specify also some specific jumps in random times and to characterize them separately from the remaining jumps of the process. In this case, the point characteristics can be presented as:

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t), \mu(\omega,t), \Gamma(\omega,t)), t \geq 0\}$$

where the random times of the specific jumps are represented by the point process $\{\Theta(\omega,t), t \geq 0\}$.

It is also possible to combine specific jumps in specific random times with jumps in deterministic moments. In this case, the parameters of the process (in the case when local characteristics exist) are:

$$\{(\Theta_1, \Theta_2(\omega,t), K_1(\omega,t), K_2(\omega,t), A(\omega,t) = \int_0^t \sigma^2(\omega,s)ds,$$

$$\mu(\omega,t,G) = \int_0^t \nu(\omega,s,G)ds + \sum_{s\in\Theta_1\cap[0,t]} K_1(\omega,s,G\setminus\{0\}) + \sum_{\theta_i<t} K_2(\omega,\theta_i,G\setminus\{0\}),$$

$$\Gamma(t) = \int_0^t \gamma(s)ds + \sum_{s\in\Theta_1\cap[0,t]} \int xK_1(\omega,s,dx) + \sum_{\theta_i<t} \int xK_2(\omega,\theta_i,dx)), t \geq 0\}.$$

Here $K_1(\omega,t)$ and $K_2(\omega,t)$ are the conditional distributions of the jumps in deterministic and specific random times correspondingly.

2. Switch Time distributions (Stoynov distributions) - STF(n,β)

We say that a random variable ξ with probability mass function $f_\xi(x)$ has distribution of $ST(n,\beta)$ family and denote this fact $\xi \in STF(n,\beta)$, if the probability mass function of ξ is given by the formula

$$f_\xi(x) = \begin{cases} \sum_{k=1}^{n+1} P(D^n = k) f_\xi(x \mid D^n = k) = \sum_{k=1}^{n+1} P(D^n = k) f_{G^k}(x), x \geq 0 \\ 0, x < 0. \end{cases}$$

where G^k are random variables with probability mass function $f_{G^k}(x) = f(k,\beta)$ and D^n are random variables taking values $k = 1,\ldots,(n+1)$ with probabilities $P(D^n = k) = \dfrac{C(n,\beta)n!}{\beta^k (n-k+1)!}, k = 1,\ldots,(n+1)$ where the coefficients $C(n,\beta)$ are given by the formulas:

$$C(n,\beta) = \frac{1}{I(n,\beta)},$$

$$I(0,\beta) = \frac{1}{\beta},$$

$$I(n,\beta) = \frac{1}{\beta} + \frac{n}{\beta} I(n-1,\beta), n = 1,2,\ldots$$

Also, variables $\tilde{D}^n = D^n - 1$ can be introduced taking values $k = 0,\ldots,n$ with probabilities

$$P(\tilde{D}^n = k) = \frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!}, k = 0,\ldots,n.$$

Then the probability mass function $f_\xi(x)$ of ξ may be presented also by the formula

$$f_\xi(x) = \begin{cases} \sum_{k=0}^{n} P(\tilde{D}^n = k) f_\xi(x \mid \tilde{D}^n = k) = \sum_{k=0}^{n} P(\tilde{D}^n = k) f_{G^{k+1}}(x), x \geq 0 \\ 0, x < 0. \end{cases}$$

Here for G^k we may adopt different families of distribution. For example, if we take $G^k \in \Gamma(k, \frac{1}{\beta}) \equiv Erlang(k, \frac{1}{\beta}),$ i. e. $\xi \mid D^n \equiv Erlang(D^n, \frac{1}{\beta}),$ we obtain a distribution which we will call $ST1(n,\beta)$ distribution, having probability density function

$$f_\xi(x) = \begin{cases} C(n,\beta)e^{-\beta x}(1+x)^n, \ x \geq 0 \\ 0, x < 0 \end{cases}.$$

Indeed, we have:

$$C(n,\beta)e^{-\beta x}(1+x)^n = C(n,\beta)e^{-\beta x}\sum_{k=0}^{n}\binom{n}{k}x^k = \sum_{k=0}^{n}C(n,\beta)\binom{n}{k}x^k e^{-\beta x} =$$
$$= \sum_{k=0}^{n}\frac{C(n,\beta)n!x^k e^{-\beta x}}{k!(n-k)!} = \sum_{k=0}^{n}\frac{C(n,\beta)n!\beta^{k+1}x^k e^{-\beta x}}{\beta^{k+1}(n-k)!k!} =$$
$$= \sum_{k=0}^{n}\frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!}\frac{\beta^{k+1}x^k e^{-\beta x}}{k!} =$$
$$= \sum_{k=0}^{n}P(D^n = k+1)f_{G^{k+1}}(x) =$$
$$= \sum_{k=1}^{n+1}\frac{C(n,\beta)n!\beta^k x^{k-1}e^{-\beta x}}{\beta^k(n-k+1)!(k-1)!} =$$
$$= \sum_{k=1}^{n+1}\frac{C(n,\beta)n!}{\beta^k(n-k+1)!}\frac{\beta^k x^{k-1}e^{-\beta x}}{\Gamma(k)} = \sum_{k=1}^{n+1}\frac{C(n,\beta)n!}{\beta^k(n-k+1)!}\frac{\beta^k x^{k-1}e^{-\beta x}}{(k-1)!} =$$
$$\sum_{k=1}^{n+1}P(D^n = k)f_{G^k}(x).$$

Special cases of $ST1(n,\beta)$ distribution are $ST1(0,\beta) \equiv Exp(\beta)$ and $ST1(1,\beta) \equiv Lindley(\beta).$ The case $n = 2$ is introduced in Stoynov (2010) and further generalized in some other publications of the author, for example Stoynov (2011) where Moment Generating Function of $ST1(2,\beta)$ is derived.

As another example we may consider negative binomial distribution for G^k, i. e. $G^k \in NB(k, e^{-\beta}).$ In this case, $\xi \mid D^n \equiv NB(D^n, e^{-\beta})$ and we obtain a distribution which we will call $ST2(n,\beta)$ distribution.

Special cases of $ST2(n,\beta)$ distribution are $ST2(0,\beta) \equiv NB(1,e^{-\beta})$ and $ST2(1,\beta) \equiv \frac{\beta}{\beta+1}NB(1,e^{-\beta}) + \frac{1}{\beta+1}NB(2,e^{-\beta}).$

We say that a random variable ξ has a negative binomial distribution (Pólya distribution) and write this $\xi \in NB(r,p)$, if its probability satisfies

$$P(\xi = k) = \binom{r+k-1}{k}p^r(1-p)^k, \ k = 0,1,2,\ldots$$

We may interpret this random variable as a counter of the failures till the $r-th$ success in a sequence of Bernoulli trials.

As a third example we may consider $G^k \equiv \delta_k(x)$ - random variable which takes value k with probability one. Then $\xi \equiv D^n \in D^n(\beta).$ In this case, we say that random variable ξ has $ST3(n,\beta)$ distribution and denote $\xi \in ST3(n,\beta).$

Special cases of $ST3(n,\beta)$ distribution are $ST3(0,\beta) \equiv D^0 \equiv 1$ and $ST3(1,\beta) \equiv D^1.$

3. ST(n,β) processes.

We say that a process $X(t)$ is $ST1(n,\beta)$ ($ST2(n,\beta)$) and denote this fact $X(t) \in ST1(t;n,\beta)$ ($X(t) \in ST1(t;n,\beta)$), if it has the following properties:

1) $X(0) = 0.$

2) The process has independent increments, i. e. for

$$0 < t_1 < t_2 < \cdots < t_n$$

the random variables

$$X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent.

3) The process is continuous stochastically (continuous in probability), i. e.

$$\lim_{t \to s} X(t) = X(s),$$

where the limit is taken in probability.

4) $X(t)$ is pure jump process with jumps at times $T_{i'}, i = 1,2,\ldots$ and jump sizes $\Delta X(T_i) = 1.$

5) The intervals between two jumps are

$$\tau = T_i - T_{i-1} \in ST1(n-1,\beta)\,(ST2(n-1,\beta)), i = 0,1,\ldots, T_0 = 0$$

The first three conditions mean that $ST1(n,\beta)$ ($ST2(n-1,\beta)$) process is an additive process. If we replace condition 4) with condition

4') $X(t)$ is pure jump process with jumps at times $T_{i'}, i = 1,2,\ldots$ and jump sizes $\Delta X(T_i) = Y_i$, where Y_i are independent and identically distributed random variables,

we say that $X(t)$ is compound $ST1(n,\beta)$ ($ST2(n,\beta)$) process.

It is interesting to see for $ST(n,\beta)$ processes the distribution of the number of jumps till time t which we denote $N^n(t).$

We start with $ST1(n,\beta)$ process. When $X(t) \in ST1(t;n,\beta)$, we consider first the case $X(t) \in ST1(t;1,\beta)$. In this case, $\tau = T_i - T_{i-1} \in ST1(0,\beta) \equiv Exp(\beta), i = 0,1,\ldots, T_0 = 0$. In this case the process $N^1(t)$ is a Poisson process $N(t)$ and for it we have

$$P(N^1(t) = k) = \frac{(\beta t)^k}{k!} e^{-\beta t}.$$

In the case when $X(t) \in ST1(t;n,\beta)$ the intervals between two jumps are

$$\tau = T_i - T_{i-1} \in ST1(n-1,\beta) \equiv Erlang(D^n,\beta), i = 0,1,\ldots, T_0 = 0.$$

First we will consider the case where

$$\tau = T_i - T_{i-1} \in Erlang(n,\beta), i = 0,1,\ldots, T_0 = 0.$$

In this case, we have that the process which counts the jumps of an Erlang process till time t which we denote $N^E(t)$ is a Poisson process $N(t)$ with omitted $n-1$ jumps and actual jumps at every $n-$th jump of the Poison process, so

$$P(N^E(t) = k) = P(N(t) = kn) = \frac{(\beta t)^{kn}}{(kn)!} e^{-\beta t}.$$

So, in the case $X(t) \in ST1(t;n,\beta)$ we have

$$P(N^n(t) = k) = P(N(t) = kn) = \frac{(\beta t)^{kn}}{(kn)!} e^{-\beta t}.$$

4. Simulation of ST(n,β) distribution.

The simulation of $ST(n,\beta)$ distribution and corresponding PMF graphics for different values of the parameters n and β can be done by using R language. Here is the code for creating graphic representation of the PMF (for $n = 100$ and $\beta = 2$):

```
#ST(n,beta) graphics
beta<-2
n<-100
In_beta<-function(n,beta, recursive=TRUE)
```

```
{if (n==0) return (1/beta) else return (1/beta+n/beta*In_beta(n-1,beta))}
Cn_beta<-function(n,beta) {1/In_beta(n,beta)}
STn_beta<-function(x,n,beta){Cn_beta(n,beta)*(1+x)^n*exp(-x*beta)}
m<-10000
STn_beta_values<-0:m
T<-100
tvec<-T*0:m/m
for (i in 1:m) {
	STn_beta_values[i]<-STn_beta(i*T/m,n,beta)
}
freq<-10   # take out every 1000th point in the simulation & plot it
pick<-freq*(0:(m/freq))+1
title<-paste("ST(", round(n,1), ",",round(beta,1), ")")
plot(tvec[pick],STn_beta_values[pick],type='l',xlab="x",ylab="f(x)",main=title,ylim=c(0,0.2))
```

The graphics for the case $n=100$ and $\beta=2$ is given in Fig 1. The graphics for the case $n=40$ and $\beta=2$ is given in Fig 2.

5. Simulation of ST(n,β) processes.

To make simulation of $ST(n,\beta)$ process, the following algorithm can be applied:

1. Define interval $[0,T]$ of the simulation.
2. Set $k=0$.
3. While $\sum_{i=1}^{k}\tau_i < T$ do:

    3.1. Set $k=k+1$

    3.2. Generate $\tau_k \in ST(n-1,\beta)$.

    3.3. Set $Y_k=1$ for standard $ST(n,\beta)$ process or simulate Y based on a given distribution f for compound $ST(n,\beta)$ process.

Then the trajectory of $X(t)\in ST(t;n,\beta)$ is given by the formula

$$X(t)=\sum_{i=1}^{N(t)}Y_i,$$

where

$$N(t)=\sum_i 1_{\{T_i<T\}}.$$

Here is the code for creating graphic representation of the $ST(n,\beta)$ process:

```
#ST(n+1,beta)_process simulation
# Theta_grid_sets Poisson, Erlang, ST
T<-1000
lambda<-0.1
Theta_Poisson <-rexp(1,lambda)
while(max(Theta_Poisson)<T) Theta_Poisson <-c(Theta_Poisson,max(Theta_Poisson)+rexp(1,lambda))
m<-length(Theta_Poisson)
Theta_Poisson[m]<-T

# Erlang
beta_ER<-lambda
n_ER<-2
m_ER<-m/n_ER
Theta_ER<- Theta_Poisson[n_ER]
for (i in 2:m_ER)
{
Theta_ER<- c(Theta_ER, Theta_Poisson[i*n_ER])
}
```

```
#ST
ST_n<-5
ST_beta<-0.1
n<-20
tvec<-T*0:n/n
In_beta<-function(n,beta, recursive=TRUE)
{if (n==0) return (1/beta) else return ((1/beta)+(n/beta)*In_beta(n-1,beta))}
Cn_beta<-function(n,beta) {1/In_beta(n,beta)}
FDn_inverse<-function(y,n,beta){
Dn_values<-1:(n+1)
Dn_prob<-1:(n+1)
for (i in 1:(n+1))
{
Dn_prob[i]<-(Cn_beta(n,beta)*factorial(n)/(beta^i*factorial(n-i+1)))
}
Prob_acum<-Dn_prob[1]+Dn_prob[2]
for (k in 1:n)
{
if (y< Prob_acum) return (k)
else
Prob_acum<-Prob_acum+Dn_prob[k+2]
}
return(n+1)
}


#Dn_simulation<-function(n,ST_n,ST_beta){
Dn_values<-1:(n+1)
Dn_prob<-1:(n+1)
Dn_simulated<-1:n
for (i in 1:(n+1))
{
Dn_prob[i]<-(Cn_beta(n,ST_beta)*factorial(n)/(ST_beta^i*factorial(n-i+1)))
}
for (i in 1:n)
{
St_unif_value<- runif(1, min = 0, max = 1)
Dn_simulated[i]<-FDn_inverse(St_unif_value,ST_n,ST_beta)
}
acc_ST<- Dn_simulated[1]

Theta_ST<- Theta_Poisson[acc_ST]

while(acc_ST<m)
{
acc_ST<-(acc_ST+Dn_simulated[i])
if (acc_ST<=m)
Theta_ST<- c(Theta_ST, Theta_Poisson[acc_ST])
}
Theta_ST
s<-8000
tvec<-T*0:s/s
N<-0:s
N[1]<-0
u<-length(Theta_ST)

for (i in 1:s)
{
        N[i]<-0
```

```
		for (j in 1:u)
		{
				if ((T*(i-1)/s)> Theta_ST[j]) N[i]<-(N[i]+1)
		}
}
title<-paste("Траектория на ST процес n=",round((ST_n+1),1), ", beta=", round(ST_beta,2))
plot(tvec[1:(s-1)],N[1:(s-1)],type='l', xlab="t",ylab="N(t)", main=title)
```

The graphics of a trajectory of $ST(n,\beta)$ process for the case $n = 6$ and $\beta = 0.1$ is given in Fig 3.

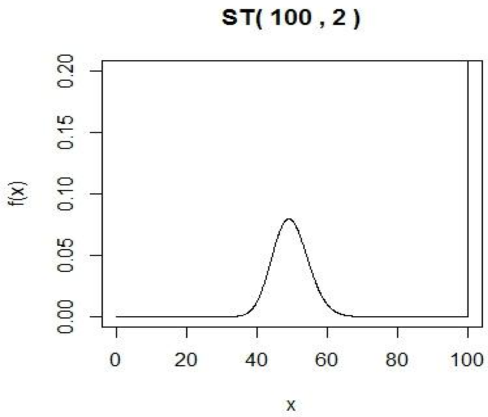

Figure 1. The graphics of $ST(n,\beta)$ distribution for the case $n = 100$ and $\beta = 2$.

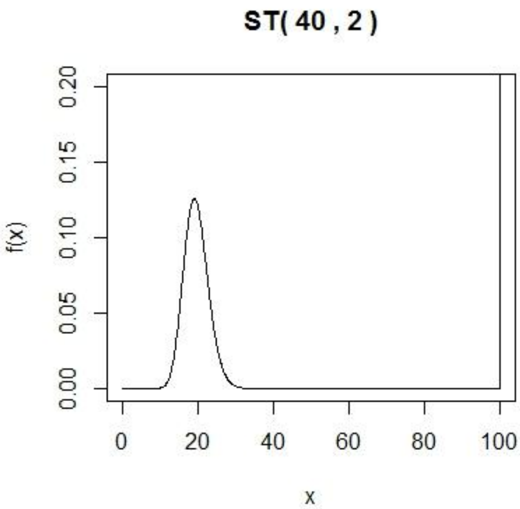

Figure 2. The graphics of $ST(n,\beta)$ distribution for the case $n = 40$ and $\beta = 2$.

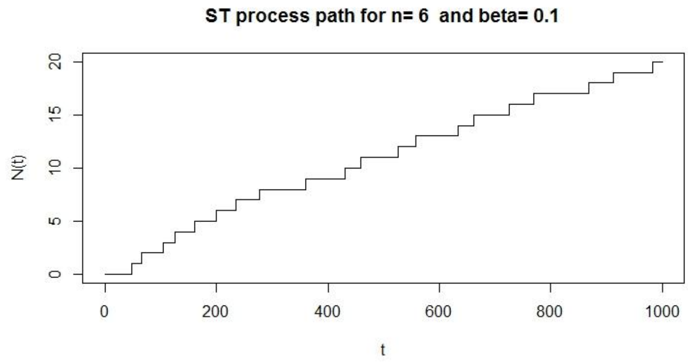

Figure 3. The graphics of a path of $ST(n,\beta)$ process for the case $n=6$ and $\beta=0.1$.

References.

Grigelionis, B. On Non-Linear Filtering Theory and Absolute Continuity of Measures, Corresponding to Stochastic Processes. Proceedings of the Second Japan-USSR Symposium of Probability Theory, Lecture Notes in Mathematics 330, 80-94. Springer, Berlin. 1973.

Kallsen, J. Semimartingale Modelling in Finance. PhD thesis, Freiburg, Universität Freiburg. 1988.

Lévy, P. Théorie de l'Addition des Variables Aléatoires. Gauthier Villars. Paris. 1934.

Lévy, P. Processus Stochastiques et Mouvement Brownien. Edition Jacques Gabay: Sceaux. 1992.

Sato, K. Lévy processes and infinitely divisible distributions. Cambridge University Press: Cambridge , UK. 1999.

Stoynov, P. Negative binomial distributions and applications. Proceedings of Third International conference "Financial and Actuarial Mathematics – FAM – 2010" (18-20.09.2010, Sofia, Bulgaria) – Part One, PERUN-SPRINT PUBLISHING, Sofia, 2010, 12-20.

Stoynov, P. Mixed Negative Binomial distribution by weighted gamma mixing distribution. - Proceedings of the Fortieth Spring conference of the Union of Bulgarian Mathematicians – Borovets, Bulgaria, 5-9 April, 2011, 327-331.