Some applications of sums of random variable in non-life insurance*

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Abstract

In the non-life insurance business, an actuary faces the problem of determining the distribution function of a sum of random variables which are not necessarily independent, like aggregate claims of an insurance portfolio. The paper points out some applications of approximating such sums when the individual distribution functions of the terms are known, but their dependence unknown.

Key words: comonotonicity, lognormal distribution, insurance premium principle, stop-loss premium.

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1.Introduction

In the actuarial domain, both life and non-life insurance, an actuary faces the problem of determining the distribution of a sum of random variables.

We deal with such sums when we consider: the total claim associated to a portfolio of risks corresponding to a certain limited interval of time, regarding damage compensations, or the accumulated fund at the end of an extended period of time (an investment problem), or the present value of a sequence of deterministic payments at successive moments of time which are meant to cover some future compulsory payments.

The assumption that in a portfolio, the risks, described as random variables, are independent is convenient (suitable) for the calculus. In reality, the random variables involved are dependent, this fact making the computations regarding the distribution of the sum quite complicated. Thus, a new problem appears: the one concerning how to find the distribution of the sum of non-independent random variables. In the case we know the individual distributions of each term, but not the stochastic dependence between them, one may determine approximations for such a sum. From an actuary point of view, these approximations can be "pessimistic" and "optimistic", reflecting his behavior characterized by aversion to risk. His preferences are modeled by convex ordering.

A special class of random variables allowing such approximations is the class of comonotonic random variables. We will introduce the notion of comonotony, we will present some theoretical results which will lead us to the conclusion that the initial aggregate loss of a portfolio is dominated by the aggregate comonotonic loss, according to the convex order. We will give a numerical illustration for a portfolio of uniformly distributed risk variables.

2. Theoretical results

The whole information on a random variable X can be found in its cumulative distribution function F_X. For an insurance company, the random variable could represent the risk associated to an insurance policy. An actuary uses various mathematical and statistical techniques in order to determine (find) the cumulative distribution function. Often, in the non-life insurance, for example, he will deal with random variables like the aggregate loss $S=\sum_{i=1}^{n} X_i$, where the terms $X_1,...,X_n$ are not mutually independent, and the distribution of the random vector $\underline{X} \overset{not}{=} (X_1,...,X_n)$ is not completely specified or much too complicated, knowing only the marginal distributions of the random variables X_i.

The insurance company is interested to propose a cautious strategy, meaning a dependency structure for a random variable $\underline{X}$, dependency which is meant to provide a larger or less favorable aggregate loss in the sense of convex order. We will use the convex order for ordering risks because, in the utility theory, for decidents with risk aversion that have to choose between risks having the same expected value, their preferences are modeled by it.

One can prove that for a random vector with given marginals, the largest sum (or aggregate loss) in the convex order sense can be obtained in the case the random vector $(X_1,...,X_n)$ has comonotonic distribution, i.e. any two possible realizations $(x_1,...,x_n)$ and $(y_1,...,y_n)$ of the random vector $(X_1,...,X_n)$ are componentwise ordered. If $\underline{x}=(x_1,...,x_n)$ and $\underline{y}=(y_1,...,y_n)$ are two vectors in the n-dimensional $\mathbf{R}^n$ real vector space, the componentwise ordering is defined as: $\underline{x}\le\underline{y}\Leftrightarrow x_i\le y_i, \forall i=\overline{1,n}$.

Definition. The set $A\subseteq R^n$ is called a comonotonic set if for any vectors $\underline{x},\underline{y}\in A$, then $\underline{x}\le\underline{y}$ or $\underline{y}\le\underline{x}$.

The support of the random vector $\underline{X}=(X_1,...,X_n)$ is the smallest set $A\subseteq R^n$ with the property $P(\underline{X}\in A)=1$.

Definition. A random vector $\underline{X}=(X_1,...,X_n)$ is comonotonic if it has a comonotonic support.

By definition, if $\underline{x}$ and $\underline{y}$ are two elements of the comonotonic support of the random vector $\underline{X}$, i.e. two possible outcomes of $\underline{X}$, then they must be ordered on components. This explains the use of the word comonotony (common monotony). A larger value of one of the components X_j of the random vector $\underline{X}$ implies a larger value of any other component X_k.

The concept of comonotony was introduced by Schmeidler (Schmeidler, D. (1986). „Integral representation without additivity", *Proceedings of the American Mathematical Society* 97, 255-261), and by Yaari (Yaari, M.E. (1987). „The dual theory of choice under risk", *Econometrica* 55, 95-115) and by Roell (Roell, A. (1987). „Risk aversion in Quiggin and Yaari's rank-order model of choice under uncertainty", *The Economic Journal* 97 (Conference 1987), 143-159).

The cumulative distribution function of the random variable X is $F_X:R\to[0,1]$, $F_X(x)=P(X\le x)$ an increasing and continuous to the right function with $F_X(-\infty)=\lim\limits_{x\to-\infty}F_X(x)=0$ and $F_X(+\infty)=\lim\limits_{x\to+\infty}F_X(x)=1$.

The inverse of the distribution function is $F_X^{-1}:[0,1]\to R$ an increasing and continuous to the left function defined by $F_X^{-1}(p)=\inf\{x\in R|F_X(x)\ge p\}, p\in[0,1]$, with the convention $\inf\phi=+\infty$. From the definition it follows that $\forall x\in\mathbf{R}$ and $\forall p\in[0,1]$, then $F_X^{-1}(p)\le x\Leftrightarrow p\le F_X(x)$.

We denote by U the random variable uniformly distributed over the interval (0,1), $U\in Uniform(0,1)$, $f_U(u)=1,\forall u\in(0,1)$, $F_U(u)=P(U\le u)=u,\forall u\in(0,1)$ and $F_U^{-1}(p)=p,\forall p\in(0,1)$.

The following theorem gives equivalent characterizations of a comonotonic random vector.

Theorem Let $\underline{X}=(X_1,...,X_n)$ be a random vector. The following are equivalent:

(1') $\underline{X}$ is a comonotonic random vector;

(1) $\underline{X}$ had a comonotonic support;

(2) For any $\underline{x}=(x_1,...,x_n)$, $F_{\underline{X}}(\underline{x})=\min\{F_{X_1}(x_1),F_{X_2}(x_2),...,F_{X_n}(x_n)\}$;

(3) For $U\in Uniform(0,1)$, we have $\underline{X}\overset{(R)}{=}\left(F_{X_1}^{-1}(U),F_{X_2}^{-1}(U),...,F_{X_n}^{-1}(U)\right)$, $\overset{(R)}{=}$ means equality in distribution;

(4) There exist a random variable *Z* and the increasing functions $f_i, i=\overline{1,n}$,

such that $\underline{X}\overset{(R)}{=}(f_1(Z),f_2(Z),...,f_n(Z))$.

Proposition $X\overset{(R)}{=}F_X^{-1}(U)$, where *U* is the random variable uniformly distributed on (0,1), $U\in Uniform(0,1)$, and $\overset{(R)}{=}$ means equality in distribution.

Theorem Let X and $g(X)$ two random variables and let $p\in(0,1)$. If g is an increasing and continuous to the left function, then $F_{g(X)}^{-1}(p)=g\left(F_X^{-1}(p)\right)$;

In the following, for any random vector $(X_1,...,X_n)$, we will denote by $(X_1^c,...,X_n^c)$ the comonotonic random vector having the same marginal distributions as the random vector $(X_1,...,X_n)$. For a random vector $\underline{X}=(X_1,...,X_n)$, a realization of the corresponding comonotonic random vector $\underline{X}^c=(X_1^c,...,X_n^c)$ lies, with probability 1, within the set $\left\{\left(F_{X_1^c}^{-1}(p),...,F_{X_n^c}^{-1}(p)\right)\middle| p\in(0,1)\right\}$, or $\left\{\left(F_{X_1}^{-1}(p),...,F_{X_n}^{-1}(p)\right)\middle| p\in(0,1)\right\}$ as $X_1\overset{(R)}{=}X_1^c,...,X_n\overset{(R)}{=}X_n^c$, where we used the fact that $\underline{X}^c=(X_1^c,...,X_n^c)\overset{(R)}{=}\left(F_{X_1^c}^{-1}(U),...,F_{X_n^c}^{-1}(U)\right)$, with $U\in Uniform(0,1)$.

For the comonotonic random vector $(X_1^c,...,X_n^c)$, we will denote $S^c=X_1^c+...+X_n^c$ the sum of its components. We will prove that approximating the distribution of the aggregate loss $S=X_1+...+X_n$ by the distribution of the comonotonic sum S^c is a prudent strategy, in the sense of the convex order, i.e. $S\le_{cx}S^c$.

Stochastic order: The concept of stochastic dominance and stop-loss order have been introduced in the actuarial science because the essence of the actuary profeesion is to compare risks.

Definition: Let *X* and *Y* be two random variables describing the losses or the risks. About them we make the assumption that they have finite means.

- *X* is said to precede *Y* in the stochastic dominance sense
  $X\le_{st}Y\Leftrightarrow F_X(x)\ge F_Y(x),\forall x\in R.$
- *X* is said to precede *Y* in the stop-loss order sense if and only if *X* has lower stop-loss premium than *Y*, i.e.
  $X\le_{sl}Y\Leftrightarrow E\left[(X-d)_+\right]\le E\left[(Y-d)_+\right],\forall d\in \mathbf{R}.$

- X is said to precede Y in the convex order sense
  $X \leq_{cx} Y \Leftrightarrow E[(X-d)_+] \leq E[(Y-d)_+], \forall d \in \mathbf{R}$ and, in addition $E[X] = E[Y]$.

The following theorem gives the inverse of the cumulative distribution function for the comonotonic sum.

Theorem The inverse of $F_{S^c}(p)$ verifies $F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), \forall p \in (0,1)$.

Proof: Let us consider the random vector $(X_1,..., X_n)$ and let $(X_1^c,..., X_n^c)$ be the corresponding comonotonic random.

As $\underline{X}^c = (X_1^c,..., X_n^c) \overset{(R)}{=} \left(F_{X_1^c}^{-1}(U),..., F_{X_n^c}^{-1}(U)\right)$,

then $S^c = X_1^c + ... + X_n^c \overset{(R)}{=} F_{X_1}^{-1}(U) + ... + F_{X_n}^{-1}(U)$, for $U \in Uniform(0,1)$.

We introduce the function $g(u) = \sum_{i=1}^{n} F_{X_i}^{-1}(u), u \in (0,1)$, which is an increasing and continuous to the left function, $S^c \overset{(R)}{=} g(U)$.

Thus $F_{S^c}^{-1}(p) = F_{g(U)}^{-1}(p) = g\left(F_U^{-1}(p)\right) = g(p), p \in (0,1)$

$\Rightarrow F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), p \in (0,1)$.

Stop-loss cover of the losses

In the insurance business, a widely used practice is to cover only partially the risks or the losses. Among the different schemes dividing the loss between insured-insurer or insurer-reinsurer, one can distinguish that the one which is often used is the stop-loss cover, where a deductible is introduced. We distinguish individual stop-loss cover on components, and respectively, the collective or aggregate stop-loss cover.

Definition. The stop-loss premium associated to the risk X, with deductible $d \in \mathbf{R}$, is defined as $E[(X-d)_+]$, where $(a-b)_+ = \begin{cases} 0, & a \leq b \\ a-b, & a > b \end{cases}$.

The next proposition gives formulas for stop-loss premiums associated to risks which follow normal, lognormal, and uniform distribution laws.

Proposition

1) Let $X \in N(\mu, \sigma^2)$ and $d \in \mathbf{R}$,

then the stop-loss premium is $E[(X-d)_+] = \mu - d + \sigma \int_{-\infty}^{\frac{d-\mu}{\sigma}} \phi(y) dy$.

2) Let $X \in Lognormal(\mu, \sigma^2)$ and $d > 0$,

then the stop-loss premium is $E[(X-d)_+] = e^{\mu + \frac{\sigma^2}{2}} \cdot (1 - \phi(\alpha - \sigma)) - d \cdot (1 - \phi(\alpha))$,

where $\alpha = \dfrac{\ln d - \mu}{\sigma}$, and $\phi(z) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt, z \in \mathbf{R}$ being the cumulative distribution function of the standard normal distribution law $N(0,1)$.

3) Let $X \in Uniform(\alpha, \beta)$, $0 < \alpha < \beta$ and $d > 0$,

then $E[(X-d)_+]=\begin{cases}\frac{\alpha+\beta}{2}-d & ,0<d\le\alpha\\ \frac{(\beta-d)^2}{2(\beta-\alpha)} & ,\alpha<d<\beta\\ 0 & ,d\ge\beta\end{cases}$.

Proof Let X be a random variable with finite expected value and let $d\in\mathbf{R}$.

$(X-d)_+=\begin{cases}0, & X\le d\\ X-d, & X>d\end{cases}$, $(d-X)_+=\begin{cases}d-X, & X\le d\\ 0, & X>d\end{cases}$

Integrating by parts, one can show that:

$$E[(X-d)_+]=\int_d^{+\infty}(x-d)dF_X(x)=$$

$$=-(x-d)(1-F_X(x))\Big|_d^{+\infty}+\int_d^{+\infty}(1-F_X(x))dx=$$

$$=\int_d^{+\infty}\overline{F_X}(x)dx,$$

Where $\overline{F_X}(x)=P(X>x)=1-P(X\le x)=1-F_X(x)$ is the survival function,

We used $\lim_{x\to\infty}(x-d)(1-F_X(x))=\lim_{x\to\infty}x(1-F_X(x))-\lim_{x\to\infty}d(1-F_X(x))$, $\lim_{x\to\infty}d(1-F_X(x))=0$.

As $E[X]<\infty$, meaning that the generalized integral $\int_{-\infty}^{+\infty}x\,dF_X(x)$ is convergent,

we have that $\int_x^{+\infty}t\,dF_X(t)\xrightarrow[x\to\infty]{}0$ ("the tails" are convergent to 0).

With this observation, we obtain that, for $x\to\infty$,

$|x(1-F_X(x))|=|x|(1-F_X(x))=x\int_x^{\infty}dF_X(t)\le\int_x^{\infty}t\,dF_X(t)\xrightarrow[x\to\infty]{}0$, that is $\lim_{x\to\infty}x(1-F_X(x))=0$.

(Comment: the stop-loss premium with deductible *d* can be thought as the weight of the upper tail of the distribution of the random variable *X*, i.e. the area of the domain which lies between the graph of the cumulative distribution function and the constant function 1, from *d* la $+\infty$.

Analogously, $E[(d-X)_+]=\int_{-\infty}^{d}(d-x)dF_X(x)=$

$$=(d-x)F_X(x)\Big|_{-\infty}^{d}+\int_{-\infty}^{d}F_X(x)dx=$$

$$=\int_{-\infty}^{d}F_X(x)dx,$$

with $\lim_{x\to-\infty}(d-x)F_X(x)=\lim_{x\to-\infty}d\,F_X(x)-\lim_{x\to-\infty}x\,F_X(x)$, $\lim_{x\to-\infty}d\,F_X(x)=d\cdot 0=0$.

As $E[X]<\infty$, then the generalized integral $\int_{-\infty}^{+\infty}x\,dF_X(x)$ is convergent, And we have that $\int_{-\infty}^{x} t\, dF_X(t) \xrightarrow[x\to-\infty]{} 0$ ("the tails" are convergent to 0).

For $x \to -\infty$, $0 \ge x F_X(x) = x \int_{-\infty}^{x} dF_X(t) \ge \int_{-\infty}^{x} t\, dF_X(t) \xrightarrow[x\to-\infty]{} 0$, that is $\lim_{x\to-\infty} x F_X(x) = 0$.

(Comment: The value $E[(d-X)_+]$ can be interpreted as the weight of the lower tail of the distribution of the random variable *X*, , i.e. the area of the domain which lies between the horizontal axes and the graph of the cumulative distribution function of *X*, from $-\infty$ up to *d*.)

One can notice that $(X-d)_+ - (d-X)_+ = X - d$

and so, in mean, we can write $E[(X-d)_+] = E[X] - d + E[(d-X)_+]$.

1) Let $X \in N(\mu, \sigma^2)$, $E[(X-d)_+] = \mu - d + E[(d-X)_+] = \mu - d + \int_{-\infty}^{d} F_X(x)dx =$

$$= \mu - d + \int_{-\infty}^{d} F_X(x)dx = \mu - d + \int_{-\infty}^{d} \phi\left(\frac{x-\mu}{\sigma}\right)dx =$$

$= \mu - d + \sigma \int_{-\infty}^{\frac{d-\mu}{\sigma}} \phi(y)dy$, a value which can be computed using numerical methods or specialized software.

2) Let $X \in Lognormal(\mu, \sigma^2)$, $Y = \ln X, Y \in N(\mu, \sigma^2)$, or $X = e^Y$.

$$E[X] = E[e^Y] = g_Y(t)\Big|_{t=1} = e^{t\mu + \frac{t^2\sigma^2}{2}}\Big|_{t=1} = e^{\mu + \frac{\sigma^2}{2}},$$

and, for $d > 0$, $E[(d-X)_+] = \int_0^d (d-x)dF_X(x) =$

$$= d\int_0^d dF_X(x) - \int_0^d x\, dF_X(x) =$$

$$= d \cdot F_X(d) - \int_0^d \frac{1}{\sigma\sqrt{2\pi}x} x e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} dx.$$

We consider the substitution $\frac{\ln x - \mu}{\sigma} = y, x = e^{\mu + \sigma y} \Rightarrow dx = \sigma e^{\mu + \sigma y} dy$,

x	0_+	d
$y = \frac{\ln x - \mu}{\sigma}$	$-\infty$	$\frac{\ln d - \mu}{\sigma} \overset{not}{=} \alpha$

$$\Rightarrow E[(d-X)_+] = d \cdot P(X \le d) - \frac{1}{\sqrt{2\pi}} e^{\mu + \frac{1}{2}\sigma^2} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}(y-\sigma)^2} dy$$

Taking $t = y - \sigma \Rightarrow y = t + \sigma, dy = dt,$

y	$-\infty$	α
$t = y - \sigma$	$-\infty$	$\alpha - \sigma$

$$\Rightarrow E\left[(d-X)_+\right] = d\cdot P\left(e^Y \le d\right) - \frac{1}{\sqrt{2\pi}} e^{\mu+\frac{1}{2}\sigma^2} \int_{-\infty}^{\alpha-\sigma} e^{-\frac{1}{2}t^2} dt =$$

$$= d\cdot P(Y \le \ln d) - e^{\mu+\frac{1}{2}\sigma^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha-\sigma} e^{-\frac{1}{2}t^2} dt =$$

$$= d\cdot P\left(\frac{Y-\mu}{\sigma} \le \frac{\ln d-\mu}{\sigma}\right) - e^{\mu+\frac{1}{2}\sigma^2} \cdot \phi(\alpha-\sigma) =$$

$$= d\cdot \phi(\alpha) - e^{\mu+\frac{1}{2}\sigma^2} \cdot \phi(\alpha-\sigma).$$

Finally, $E\left[(X-d)_+\right] = E[X] - d + E\left[(d-X)_+\right] =$

$$= e^{\mu+\frac{\sigma^2}{2}} - d + d\cdot\phi(\alpha) - e^{\mu+\frac{1}{2}\sigma^2}\cdot\phi(\alpha-\sigma) =$$

$$= e^{\mu+\frac{\sigma^2}{2}} \cdot (1-\phi(\alpha-\sigma)) - d\cdot(1-\phi(\alpha)), \text{ where } \alpha = \frac{\ln d-\mu}{\sigma}.$$

3) $(X-d)_+ = \begin{cases} X-d, & X \ge d \\ 0, & X < d \end{cases}$

If $0 < d \le \alpha \Rightarrow X \ge d \Rightarrow (X-d)_+ = X-d$

$$\Rightarrow E\left[(X-d)_+\right] = E[X-d] = \int_\alpha^\beta x\frac{1}{\beta-\alpha}dx - d = \frac{\alpha+\beta}{2} - d.$$

If $\alpha < d < \beta \Rightarrow (X-d)_+ = \max\{X-d, 0\} = \begin{cases} 0 & ,\alpha < X < d \\ X-d & ,d \le X < \beta \end{cases}$

$$\Rightarrow E\left[(X-d)_+\right] = \int_d^\beta (x-d)\frac{1}{\beta-\alpha}dx = \frac{(\beta-d)^2}{2(\beta-\alpha)}.$$

In the following tables we give some values of the stop-loss premiums in the case the risks follow a normal, lognormal or uniform distribution law.

1) If $X \in N(\mu, \sigma^2)$, we get

μ	σ	d	Stop-loss premium
1500	3	150(=10%* μ)	1350
1500	30	150(=10%* μ)	1350
1500	300	150(=10%* μ)	1350
1500	300	300(=20%* μ)	1200
1500000	3	150000(=10%* μ)	1350000
1500000	4000	150000(=10%* μ)	1350000
1500000	3000	375000(=25%* μ)	1125000

2) If $X \in Lognormal(\mu, \sigma^2)$, we get

μ	σ	mean	d	Stop-loss premium
7	0.1	1102.1	150	952.13
7	0.5	1242.6	150	1092.6
7	0.8	1510.2	150	1360.4

The relationship between the individual premiums and the collective premium is given in the next theorem.

Theorem The stop-loss premium corresponding to the aggregate comonotonic loss S^c for the random vector $\left(X_1^c,...,X_n^c\right)$ is $E[(S^c-d)_+]=\sum_{i=1}^{n}[(X_i-d_i)_+]$, with $F_{S^c}^{-1}(0)<d<F_{S^c}^{-1}(1)$, where $d_i=F_{X_i}^{-1}\left(F_{S^c}(d)\right), i=\overline{1,n}$.

Using the above theorem, one can prove the next one.

Theorem (comonotonic upper bound for a sum of random variables)

For any random vector $\left(X_1,...,X_n\right)$, $X_1+X_2+...+X_n \leq_{cx} X_1^c+X_2^c+...+X_n^c$ holds true.

Observation: The above theorem states that the least attractive random vector $\left(X_1,...,X_n\right)$ having the marginal distribution functions $F_1 \overset{not}{=} F_{X_1}, F_2 \overset{not}{=} F_{X_2},...,F_n \overset{not}{=} F_{X_n}$ known, i.e. the sum of its components is the largest in the sense of convex order, has a joint comonotonic distribution, i.e. its common distribution is given by $\left(F_{X_1}^{-1}(U),F_{X_2}^{-1}(U),...,F_{X_n}^{-1}(U)\right)$. The components of this vector are dependent in a maximal way, because its components are expressed as functions of the same random variable.

2. Numerical illustration

Let the random vector $\underline{X}=\left(X_1,...,X_n\right)$, $X_i \in Uniform\left(\alpha_i,\beta_i\right), \alpha_i<\beta_i, i=\overline{1,n}$.

We can write $X_i \overset{(R)}{=} \alpha_i+\left(\beta_i-\alpha_i\right)U, \forall i=\overline{1,n}$

$\Rightarrow$ there exist the random variable $U \in Uniform(0,1)$

and there exist the strictly increasing functions $f_i(u)=\alpha_i+\left(\beta_i-\alpha_i\right)u, i=\overline{1,n}$.

$$F_{X_i}(x)=\begin{cases} 0 & ,x\leq\alpha_i \\ \dfrac{x-\alpha_i}{\beta_i-\alpha_i} & ,\alpha_i<x<\beta_i \\ 1 & ,x\geq\beta_i \end{cases} \text{ and } F_{X_i}^{-1}(p)=\alpha_i+\left(\beta_i-\alpha_i\right)p,\ p\in(0,1), i=\overline{1,n}.$$

$$\Rightarrow S^c = X_1^c+...+X_n^c \overset{(R)}{=} F_{X_1}^{-1}(U)+...+F_{X_n}^{-1}(U)=$$

$$=\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\left(\beta_i-\alpha_i\right)U,$$

thus $S^c \in Uniform\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\beta_i\right)$ and $S\leq_{cx} S^c$.

Let us consider a portfolio of non-independent $n=100$ risks $\underline{X}=\left(X_1,...,X_n\right)$ such that:

$X_1,...,X_{n_1}$, with $n_1=35$ are identical distributed with $Y_1\in Uniform\left(\alpha_1,\beta_1\right)$, $\left(\alpha_1,\beta_1\right)=(10,23)$,

$X_{n_1+1},\dots,X_{n_1+n_2}$, with $n_2 = 45$, are identical distributed with $Y_2 \in Uniform(\alpha_2, \beta_2)$, $(\alpha_2, \beta_2) = (8,17)$,
and $X_{n_1+n_2+1},\dots,X_n$, with $n_3 = 20$, are identical distributed with $Y_3 \in Uniform(\alpha_3, \beta_3)$, $(\alpha_3, \beta_3) = (13,25)$,

The comonotonic sum $S^c = X_1^c + \dots + X_n^c \overset{(R)}{=} F_{X_1}^{-1}(U) + \dots + F_{X_n}^{-1}(U) = n_1Y_1 + n_2Y_2 + n_3Y_3 =$

$$= \sum_{k=1}^{3} n_k\alpha_k + \sum_{k=1}^{3} n_k(\beta_k - \alpha_k)U$$

$\Rightarrow S^c \in Uniform\left(\sum_{k=1}^{3} n_k\alpha_k, \sum_{k=1}^{3} n_k\beta_k\right)$. i.e. $S^c \in Uniform(\alpha = 970, \beta = 2070)$,

and the aggregate loss $X_1 + X_2 + \dots + X_n \le_{cx} S^c$, so the stop-loss premiums verify the inequality $E[(X_1 + X_2 + \dots + X_n - d)_+] \le E[(S^c - d)_+]$, for any deductible *d*.

Let $d = 1200 \in (970,2070)$, then the stop-loss premium for the aggregate conomotonic loss of this portfolio is $E[(S^c - d)_+] = \dfrac{(\beta - d)^2}{2(\beta - \alpha)} = 344.0454545$.

$E[(S^c - d)_+] = \sum_{i=1}^{n} E[(X_i - d_i)_+] = \sum_{k=1}^{3} n_k E[(Y_k - d_k)_+]$, where the deductibles are found from $d_k = F_{Y_k}^{-1}(F_{S^c}(d)), k = 1,2,3$.

For $d \in (\alpha, \beta)$, $F_{S^c}(d) = \dfrac{d - \alpha}{\beta - \alpha} \in (0,1)$ and $F_{Y_k}^{-1}(p) = \alpha_k + (\beta_k - \alpha_k)p,\ p \in (0,1),\ k = 1,2,3$

$$\Rightarrow d_k = \alpha_k + (\beta_k - \alpha_k)\frac{d - \alpha}{\beta - \alpha},\ k = 1,2,3,$$

So $d_1 = 12.71818182$, $d_2 = 9.881818182$, $d_3 = 15.50909091$ and one may check that $n_1d_1 + n_2d_2 + n_3d_3 = 1200 = d$.

Finally, we can compute the individual stop-loss premiums.

For $k = \overline{1,35}$, $E[(X_k - d_1)_+] = E[(Y_1 - d_1)_+] = \dfrac{(\beta_1 - d_1)^2}{2(\beta_1 - \alpha_1)} = 4.065991734$,

for $j = \overline{36,80}$, $E[(X_j - d_2)_+] = E[(Y_2 - d_2)_+] = \dfrac{(\beta_2 - d_2)^2}{2(\beta_2 - \alpha_2)} = 2.814917355$,

and for $l = \overline{81,100}$, $E[(X_l - d_3)_+] = E[(Y_3 - d_3)_+] = \dfrac{(\beta_3 - d_3)^2}{2(\beta_3 - \alpha_3)} = 3.75322314$.

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